ON THE MORSE–BOTT PROPERTY OF ANALYTIC FUNCTIONS ON BANACH SPACES WITH ŁOJASIEWICZ EXPONENT ONE HALF

PAUL M. N. FEEHAN

Abstract. It is a consequence of the Morse–Bott Lemma (see Theorems 2.10 and 2.14) that a C^2 Morse–Bott function on an open neighborhood of a critical point in a Banach space obeys a Łojasiewicz gradient inequality with the optimal exponent one half. In this article we prove converses (Theorems 1, 2, and Corollary 3) for analytic functions on Banach spaces: If the Łojasiewicz exponent of an analytic function is equal to one half at a critical point, then the function is Morse–Bott and thus its critical set nearby is an analytic submanifold. The main ingredients in our proofs are the Łojasiewicz gradient inequality for an analytic function on a finite-dimensional vector space [58] and the Morse Lemma (Theorems 4 and 5) for functions on Banach spaces with degenerate critical points that generalize previous versions in the literature, and which we also use to give streamlined proofs of the Łojasiewicz–Simon gradient inequalities for analytic functions on Banach spaces (Theorems 8 and 9).

Contents

Date: This version: March 30, 2018.

2010 *Mathematics Subject Classification.* Primary 32B20, 32C05, 32C18, 32C25, 58E05; secondary 14E15, 32S45, 57R45, 58A07, 58A35.

Key words and phrases. Analytic varieties, Łojasiewicz inequalities, gradient flow, Morse theory, Morse–Bott functions, resolution of singularities, semianalytic sets and subanalytic sets, singularities.

The author was partially supported by National Science Foundation grant DMS-1510064 and the Dublin Institute for Advanced Studies.

1. Introduction

Let $\mathbb{K}$ be $\mathbb{R}$ or $\mathbb{C}$, and $d \geq 1$ be an integer, and let $\mathbb{K}^{d*} = (\mathbb{K}^d)^*$ denote the dual space. In order to better motivate our main results (Theorems 1, 2, and Corollary 3), we begin by recalling the well-known

Theorem 1.1 (Łojasiewicz gradient inequality for an analytic function)**.** *Let $U \subset \mathbb{K}^d$ be an open neighborhood of the origin and $f : U \to \mathbb{K}$ be an analytic function. If $f(0) = 0$ and $f'(0)) = 0$ then, after possibly shrinking U, there are constants $C \in (0, \infty)$ and $\theta \in [1/2, 1)$ such that*

$$\|f'(x)\|_{\mathbb{K}^{d*}} \geq C|f(x)|^\theta, \quad \forall\, x \in U. \tag{1.1}$$

Łojasiewicz used the theory of semianalytic sets to prove Theorem 1.1 as[1] [58, Proposition 1, p. 92 (67)] when $\mathbb{K} = \mathbb{R}$ and gave the range for θ as the interval $(0, 1)$. His article remained unpublished, but Bierstone and Milman gave a simplified and streamlined exposition of Łojasiewicz's method in [10] for $\mathbb{K} = \mathbb{R}$ and later gave an elegant and entirely new proof in [11] of (1.1) using resolution of singularities for analytic sets [42] over arbitrary fields $\mathbb{K}$ as above and for which they also gave a new and significantly simplified proof. In [10, 11], Bierstone and Milman state the range as for θ as the interval $(0, 1)$.

In [31], we proved Theorem 1.1 as [31, Theorem 1] by also appealing to resolution of singularities for analytic sets but in a different way from that of Bierstone and Milman [11] and that approach allowed us to give the forthcoming partial identification (1.4) of the Łojasiewicz exponent, θ, and show that it is restricted to the interval $[1/2, 1)$, sharpening the range $(0, 1)$ provided in [10, 11, 58]. Resolution of singularities for analytic varieties yields the following special case of [31, Theorem 4.5] (see [31, Sections 4.3 and 4.4] for details of references to statements and proofs):

[1]The first page number refers to the version of Łojasiewicz's original manuscript mimeographed by IHES while the page number in parentheses refers to the cited LaTeX version of his manuscript prepared by M. Coste and available on the Internet.

Theorem 1.2 (Monomialization of an analytic function)**.** *Let $U \subset \mathbb{K}^d$ be an open neighborhood of the origin and $f : U \to \mathbb{K}$ be an analytic function. If $f(0) = 0$ then, after possibly shrinking U, there are an open neighborhood $V \subset \mathbb{K}^d$ of the origin and an analytic map,*

$$\pi : V \ni y \mapsto x \in U, \tag{1.2}$$

such that $\pi(0) = 0$ and π restricts to an analytic diffeomorphism on the complement of the zero set, $Z := f^{-1}(0)$,

$$\pi : V \setminus \pi^{-1}(Z) \cong U \setminus Z,$$

and $\pi^* f$ is a simple normal crossing function, that is,

$$\pi^* f(y) = y_1^{n_1} y_2^{n_2} \cdots y_d^{n_d}, \quad \forall\, y \in V, \tag{1.3}$$

where the n_i are non-negative integers for $i = 1, \dots, d$.

The Łojasiewicz exponent of a monomial function can easily be computed exactly using the Generalized Young Inequality (see [31, Remark 3.1]) to give the

Lemma 1.3 (Łojasiewicz exponent of a monomial function)**.** *(See Feehan* [31, Theorem 5] *or Haraux* [39, Theorem 3.1]*.) Let $g : \mathbb{K}^d \to \mathbb{K}$ be an analytic function given by $g(y) = y_1^{n_1} y_2^{n_2} \cdots y_d^{n_d}$ for $y \in \mathbb{K}^d$, where the n_i are non-negative integers for $i = 1, \dots, d$. If $g(0) = 0$ and $g'(0) = 0$, then g obeys the Łojasiewicz gradient inequality* (1.1) *on $U = \mathbb{K}^d$ for a constant $C \in (0, \infty)$ and exponent*

$$\theta = 1 - \frac{1}{N} \in [1/2, 1), \quad \textit{where } N := \sum_{i=1}^{d} n_i \geq 2. \tag{1.4}$$

We apply Theorem 1.2 and Lemma 1.3 to prove Theorem 1.1 using the elementary

Lemma 1.4 (Łojasiewicz exponents and maps)**.** *Let d, e be positive integers, $V \subset \mathbb{K}^e$ and $U \subset \mathbb{K}^d$ be open neighborhoods of the origins and $\phi : V \to U$ be an open C^1 map such that $\phi(0) = 0$. If $f : U \to \mathbb{K}$ is a C^1 function such that $\phi^* f$ obeys the Łojasiewicz gradient inequality* (1.1) *at the origin with exponent $\theta \geq 0$ then, after possibly shrinking U, the function f obeys the Łojasiewicz gradient inequality* (1.1) *with the same exponent θ and a possibly smaller constant $C \in (0, \infty)$.*

Theorem 1.1 now follows as an immediate corollary of Theorem 1.2 and Lemmas 1.3 and 1.4. The exponent $\theta = 1/2$ is optimal in the sense that when a solution $x(t)$, for $t \in [0, \infty)$, to the negative gradient flow defined by f,

$$\frac{dx}{dt} = -\operatorname{grad} f(x(t)), \quad x(0) = x_0 \in U,$$

converges to a point $x_\infty \in \operatorname{Crit} f$ as $t \to \infty$, the norm of the difference, $\|x(t) - x_\infty\|_{\mathbb{K}^d}$, converges to zero as $t \to \infty$ like $\exp(-ct)$ for some $c > 0$ when $\theta = 1/2$ but only like $t^{-\gamma}$ for some $\gamma > 0$ when $\theta \in (1/2, 1)$: see Appendix A for a discussion and references.

The optimal exponent, $\theta = 1/2$, is achieved when $f : U \to \mathbb{K}$ is a C^2 function that is Morse–Bott at the origin, that is, the critical set, $\operatorname{Crit} f := \{x \in U : f'(x) = 0\}$, is a connected, smooth submanifold of U (after possibly shrinking U) of dimension equal to $\dim \operatorname{Ker} f''(0)$. This is readily seen by applying the Morse–Bott Lemma (see Theorems 2.10 or 2.14) to produce (after possibly shrinking U) a C^2 diffeomorphism, $\Phi : V \to U$, from an open neighborhood $V \subset \mathbb{K}^d$ of the origin onto U such that $\Phi(0) = 0$ and

$$\Phi^* f(y) = \sum_{i=1}^{d-c} a_i y_i^2, \quad \forall\, y \in V,$$

where $c = \dim \operatorname{Ker} f''(0)$ and $a_i \in \mathbb{K} \setminus \{0\}$ for $i = 1, \dots, d-c$. The Łojasiewicz gradient inequality (1.1) for f with exponent $\theta = 1/2$ follows immediately by direct calculation for $\Phi^* f$ and invariance of the Łojasiewicz exponent under diffeomorphisms. Proofs of the optimal Łojasiewicz gradient inequality for Morse–Bott functions on $\mathbb{K}^d$ or Banach spaces over $\mathbb{K}$ were provided by the author in [31, Theorem 3] and [30, Theorem 3] and by the author and Maridakis [32, Theorems 3 and 4] without relying on the Morse–Bott Lemma.

The main goal of this article is explore whether the converse is true:

> *If $f : U \to \mathbb{K}$ is a C^2 function that obeys the Łojasiewicz gradient inequality* (1.1) *with exponent $\theta = 1/2$, then is f Morse–Bott at the origin?*

As we shall see in Theorems 1, 2 and Corollary 3, this converse is indeed true in great generality — for a broad class of analytic functions, $f : \mathscr{X} \supset \mathscr{U} \to \mathbb{K}$, on Banach spaces, $\mathscr{X}$, over $\mathbb{K}$ and for any analytic function f when $\mathscr{X} = \mathbb{K}^d$. It is apparent from examples that $\theta \in [1/2, 1)$ provides a measure of complexity of the singularity of the critical set of f, at the origin. Theorems 1, 2 and Corollary 3 make this informal measure of complexity of the singularity precise for analytic functions on open neighborhoods of the origin in $\mathbb{K}^d$ with arbitrary $d \geq 1$ and even Banach spaces over $\mathbb{K}$: the critical set is an analytic submanifold of the expected dimension when $\theta = 1/2$.

Intuition supporting the preceding conclusion can be obtained by examining the structure of the function $\pi^* f$ in (1.3) when $N = 2$, the lowest possible total degree of the monomial. Indeed, if $N = 2$, then (after relabeling coordinates) either $n_1 = 2$ and $n_i = 0$ for all $i \geq 2$ or $n_1 = n_2 = 1$ and $n_i = 0$ for all $i \geq 3$ and thus[2]

$$\pi^* f(y) = \pm y_1^2 \quad \text{or} \quad y_1 y_2, \quad \forall\, y \in V, \tag{1.5}$$

together with

$$\pi^{-1}(f^{-1}(0)) = \{y \in V : y_1 = 0\} \quad \text{or} \quad \{y \in V : y_1 = 0 \text{ or } y_2 = 0\}.$$

In particular, the critical set of $\pi^* f$ is either the codimension-one submanifold, $\{y \in V : y_1 = 0\}$, or the codimension-two submanifold, $\{y \in V : y_1 = y_2 = 0\}$. These observations tell us that the condition $\theta = 1/2$ imposes strong constraints on resolution morphism, π, and the structure of the analytic function f itself since resolution of singularities tends to 'increase degrees'.

Because the identification (1.3) of $\pi^* f$ as a simple normal crossing function comes from resolution of singularities for analytic sets, one might expect that methods of algebraic geometry could be used to compute θ directly in terms of f and also lead to the conclusion that f must be Morse–Bott, at least when f is a polynomial and possibly even when f is analytic. However, while the Łojasiewicz exponent has been estimated for certain classes of polynomials (see [31, Section 1] for a survey), it appears difficult to estimate the exponent in any generality, even for polynomial functions. Using the fact that $\pi^* f(y) = \pm y_1^2$ or $y_1 y_2$ when f has Łojasiewicz exponent $1/2$ to directly constrain the structure of f and the resolution morphism π in the proof of resolution of singularities appears challenging, although this may provide one route to a proof Corollary 3 using methods of algebraic geometry.

Our approach to proving Theorems 1, 2 in this article is analytic and relies on a version[3] (see Section 1.2) of the Morse Lemma for *analytic* functions f with *degenerate* critical points, together with our identification (1.5) of $\pi^* f$ when f has Łojasiewicz exponent $1/2$ and π is a resolution of singularities (1.2) for the zero set, $f^{-1}(0)$.

[2]We omit the pair of possible signs, $\pm$, when $\mathbb{K} = \mathbb{C}$.

[3]I am indebted to Michael Greenblatt and András Némethi for pointing out to me that this should be a key analytical tool.

The concept of a Morse–Bott function was introduced by Bott in [13, Definition, p. 248] and used by him in his first proof of the Bott Periodicity Theorem [14]. Morse–Bott functions were employed by Austin and Braam [7, Section 3] in their approach to developing a Morse theory approach to equivariant cohomology.

1.1. **Morse–Bott property of analytic functions with Łojasiewicz exponent one half.** Let $\mathscr{X}, \mathscr{Y}$ be Banach spaces over $\mathbb{K}$, and $\mathscr{L}(\mathscr{X}, \mathscr{Y})$ denote the Banach space of bounded linear operators from $\mathscr{X}$ to $\mathscr{Y}$, and $\operatorname{Ker} A$ and $\operatorname{Ran} A$ denote the kernel and range of $A \in \mathscr{L}(\mathscr{X}, \mathscr{Y})$, and $\mathscr{X}^*$ denote the continuous dual space of $\mathscr{X}$. Let $\mathscr{L}_{\mathrm{sym}}(\mathscr{X}, \mathscr{X}^*) \subset \mathscr{L}(\mathscr{X}, \mathscr{X}^*)$ denote the closed subspace of operators, A, that are symmetric in the sense that $\langle x, Ay\rangle = \langle y, Ax\rangle$ for all $x, y \in \mathscr{X}$, where $\langle \cdot, \cdot \rangle$ denotes the canonical pairing, $\mathscr{X} \times \mathscr{X}^* \ni (x, \alpha) \mapsto \alpha(x) \in \mathbb{K}$. We recall the canonical identifications,

$$\mathscr{L}_{\mathrm{sym}}(\mathscr{X}, \mathscr{X}^*) = \mathscr{L}^2_{\mathrm{sym}}(\mathscr{X}, \mathbb{K}) = \mathscr{L}_{\mathrm{sym}}(\mathscr{X} \otimes \mathscr{X}, \mathbb{K}),$$

where $\mathscr{L}^n(\mathscr{X}, \mathbb{K})$ (respectively, $\mathscr{L}(\otimes^n \mathscr{X}, \mathbb{K})$) is the Banach space of continuous n-linear (respectively, linear) functionals, $A : \times^n \mathscr{X} \to \mathbb{K}$ (respectively, $A : \otimes^n \mathscr{X} \to \mathbb{K}$), for integers $n \geq 1$.

If $\mathscr{U} \subset \mathscr{X}$ is an open subset, $f : \mathscr{U} \to \mathbb{K}$ is a C^2 function, and $\operatorname{Crit} f = \{x \in \mathscr{U} : f'(x) = 0\}$ is a C^2, connected submanifold of $\mathscr{U}$, then[4] the tangent space, $T_x \operatorname{Crit} f$, is contained in $\operatorname{Ker} f''(x)$, for each $x \in \operatorname{Crit} f$, where $f'(x) \in \mathscr{X}^*$ and $f''(x) \in \mathscr{L}_{\mathrm{sym}}(\mathscr{X}, \mathscr{X}^*)$.

Definition 1.5 (Morse–Bott properties)**.** Let $\mathscr{X}$ be a Banach space over $\mathbb{K}$, and $\mathscr{U} \subset \mathscr{X}$ be an open neighborhood of the origin, and $f : \mathscr{U} \to \mathbb{K}$ be a C^2 function such that $\operatorname{Crit} f$ is a C^2, connected submanifold.

(1) If $x_0 \in \operatorname{Crit} f$ and $\operatorname{Ker} f''(x_0) \subset \mathscr{X}$ has a closed complement $\mathscr{X}_0$ and $\operatorname{Ran} f''(x_0) = \mathscr{X}_0^*$, and $T_{x_0} \operatorname{Crit} f = \operatorname{Ker} f''(x_0)$, then f is *Morse–Bott at the point* x_0;
(2) If f is *Morse–Bott at each point* $x \in \operatorname{Crit} f$, then f is *Morse–Bott along* $\operatorname{Crit} f$ or a *Morse–Bott function*.

The Morse–Bott Lemma (see Theorem 2.10) implies that if f is Morse–Bott at a point, as in Definition 1.5 (1), then f is Morse–Bott along an open neighborhood of that point in $\operatorname{Crit} f$, as in Definition 1.5 (2). If $\operatorname{Crit} f$ in Definition 1.5 consists of isolated points and $f''(x_0) \in \mathscr{L}(\mathscr{X}, \mathscr{X}^*)$ is invertible for each $x_0 \in \operatorname{Crit} f$, then f is a Morse function. The finite-dimensional analogue of Definition 1.5 (2) is well-known.

Remark 1.6 (Morse–Bott functions on Euclidean space)*.* When $\mathscr{X}$ is finite-dimensional, the definition of a Morse–Bott function was given by Bott [13, Definition, p. 248], [14]. See Nicolaescu [63, Definition 2.41] for a modern exposition.

When $\mathscr{X}$ is infinite-dimensional, then one must impose hypotheses on f in addition to those of Bott in the finite-dimensional case in order to obtain a tractable version, such as Theorem 2.10, of the classical Morse–Bott Lemma (for example, Nicolaescu [63, Proposition 2.42]). Because the operator $f''(x_0) \in \mathscr{L}(\mathscr{X}, \mathscr{X}^*)$ is symmetric, Lemma 2.2 (1) implies that one always has $\operatorname{Ran} f''(x_0) \subset \mathscr{X}_0^*$ if $\mathscr{X}_0$ is a closed complement of $\operatorname{Ker} f''(x_0)$. Thus Item (1) in Definition 1.5 imposes the non-degeneracy condition $\operatorname{Ran} f''(x_0) = \mathscr{X}_0^*$, like in the finite-dimensional case, but subject to an explicit requirement that $\operatorname{Ker} f''(x_0)$ has a closed complement.

When the operator $f''(x_0) \in \mathscr{L}(\mathscr{X}, \mathscr{X}^*)$ is Fredholm with index zero, Lemma 2.2 (2) yields the non-degeneracy condition, $\operatorname{Ran} f''(x_0) = \mathscr{X}_0^*$. When the operator $f''(x_0) \in \mathscr{L}(\mathscr{X}, \mathscr{X}^*)$ is not Fredholm but $\mathscr{X}$ is reflexive and $\operatorname{Ker} f''(x_0)$ has a closed complement, Lemma 2.3 implies that

[4] For example, see the discussion prior to Theorem 2.10.

the condition that $\operatorname{Ran} f''(x_0) = \mathscr{X}_0^*$ in Item (1) of Definition 1.5 is equivalent to the condition that $\operatorname{Ran} f''(x_0) \subset \mathscr{X}^*$ be a closed subspace. We now state the main results of this article.

Theorem 1 (Morse–Bott property of an analytic function with Łojasiewicz exponent one half)**.** *Let $\mathscr{X}$ be a Banach space over $\mathbb{K}$, and $\mathscr{U} \subset \mathscr{X}$ be an open neighborhood of the origin, and $f : \mathscr{U} \to \mathbb{K}$ be a non-constant analytic function such that $f(0) = 0$ and $f'(0) = 0$ and $f''(0) \in \mathscr{L}(\mathscr{X}, \mathscr{X}^*)$ is a Fredholm operator with index zero. If there is a constant $C \in (0, \infty)$ such that, after possibly shrinking $\mathscr{U}$,*

$$\|f'(x)\|_{\mathscr{X}^*} \geq C|f(x)|^{1/2}, \quad \forall\, x \in \mathscr{U}, \tag{1.6}$$

then f is a Morse–Bott function in the sense of Definition 1.5.

Hence, Theorem 1 is a converse to the simpler Theorem 6 when f is *analytic*. The conclusion of Theorem 1 implies that (using the version of the Morse–Bott Lemma provided by Theorem 2.10), after possibly shrinking $\mathscr{U}$, there are an open neighborhood, $\mathscr{V} \subset \mathscr{X}$, of the origin and an analytic diffeomorphism, $\Phi : \mathscr{V} \to \mathscr{U}$, such that $\Phi(0) = 0$ and

$$f(\Phi(y)) = \frac{1}{2}\langle y, Ay\rangle, \quad \forall\, y \in \mathscr{V},$$

where $A = (f \circ \Phi)''(0) \in \mathscr{L}_{\mathrm{sym}}(\mathscr{X}, \mathscr{X}^*)$ and, letting $\mathscr{K} := \operatorname{Ker} A \subset \mathscr{X}$ denote the finite-dimensional kernel of A with closed complement, $\mathscr{X}_0 \subset \mathscr{X}$, and $\operatorname{Ran} A = \mathscr{X}_0^* \subset \mathscr{X}^*$ (see Lemma 2.2 (2)) denote the closed range of A with finite-dimensional complement, $\mathscr{K}^*$, in $\mathscr{X}^* = \mathscr{X}_0^* \oplus \mathscr{K}^*$ (see Lemma 2.1), we have

$$A = \begin{pmatrix} A_0 & 0 \\ 0 & 0 \end{pmatrix} : \mathscr{X}_0 \oplus \mathscr{K} \to \mathscr{X}_0^* \oplus \mathscr{K}^*,$$

where $A_0 \in \mathscr{L}_{\mathrm{sym}}(\mathscr{X}_0, \mathscr{X}_0^*)$ is an isomorphism of Banach spaces. Thus, $(f \circ \Phi)'(y) = Ay$, for all $y \in \mathscr{V}$, and $\operatorname{Crit} f \circ \Phi = \mathscr{V} \cap \operatorname{Ker} A$, an analytic submanifold of $\mathscr{V}$ of dimension equal to $\dim \operatorname{Ker} A$.

As explained in [32, Section 1.1], the hypotheses of Theorem 1 are restrictive since they imply that $\mathscr{X}$ is isomorphic to its continuous dual space, $\mathscr{X}^*$. (There are examples of Banach spaces that are isomorphic to their dual spaces but are not isomorphic to Hilbert spaces. However, even the implication that $\mathscr{X}$ is a reflexive Banach space is already restrictive for some applications of infinite-dimensional Morse Theory to geometric analysis. A classical theorem of Lindenstrauss and Tzafriri [56] asserts that a real Banach space in which every closed subspace is complemented (that is, is the range of a bounded linear projection) is isomorphic to a Hilbert space.) As in [76], one can relax the restriction that $\mathscr{X} \cong \mathscr{X}^*$ by introducing an extrinsic gradient operator, $\mathscr{M}(x)$, to represent the derivative, $f'(x)$, for each $x \in \mathscr{U}$.

Definition 1.7 (Gradient map)**.** (See Berger [9, Section 2.5] or Huang [47, Definition 2.1.1].) Let $\mathscr{X}$ and $\tilde{\mathscr{X}}$ be Banach spaces over $\mathbb{K}$, and $\tilde{\mathscr{X}} \subset \mathscr{X}^*$ be a continuous embedding, and $\mathscr{U} \subset \mathscr{X}$ be an open subset. A continuous map, $\mathscr{M} : \mathscr{U} \to \tilde{\mathscr{X}}$, is a *gradient map* if there is a C^1 function, $f : \mathscr{U} \to \mathbb{K}$, its *potential function*, such that

$$f'(x)v = \langle v, \mathscr{M}(x)\rangle, \quad \forall\, x \in \mathscr{U}, \quad v \in \mathscr{X}, \tag{1.7}$$

where $\langle \cdot, \cdot \rangle : \mathscr{X} \times \mathscr{X}^* \to \mathbb{K}$ is the canonical pairing.

A continuous embedding of Banach spaces, $\tilde{\mathscr{X}} \subset \mathscr{X}^*$, induces a continuous embedding of Banach spaces of bounded linear operators,

$$\mathscr{L}(\mathscr{X}, \tilde{\mathscr{X}}) \subset \mathscr{L}(\mathscr{X}, \mathscr{X}^*),$$

since if $T \in \mathscr{L}(\mathscr{X}, \tilde{\mathscr{X}})$, then $T \in \mathscr{L}(\mathscr{X}, \mathscr{X}^*)$ by composing T with the continuous embedding, $\tilde{\mathscr{X}} \subset \mathscr{X}^*$. We can therefore define

$$\mathscr{L}_{\mathrm{sym}}(\mathscr{X}, \tilde{\mathscr{X}}) := \mathscr{L}(\mathscr{X}, \tilde{\mathscr{X}}) \cap \mathscr{L}_{\mathrm{sym}}(\mathscr{X}, \mathscr{X}^*).$$

Some basic properties of gradient maps are listed in Proposition 2.5, including the fact that $\mathscr{M}(x) \in \mathscr{L}_{\mathrm{sym}}(\mathscr{X}, \tilde{\mathscr{X}})$ for all $x \in \mathscr{U}$. When $\tilde{\mathscr{X}} = \mathscr{X}^*$ in Definition 1.7, then the derivative and gradient maps coincide. If we are given a C^1 function, $f : \mathscr{U} \to \mathbb{K}$, such that $f'(x) = \langle \cdot, \mathscr{M}(x)\rangle$, $x \in \mathscr{U}$, for a C^0 map, $\mathscr{M} : \mathscr{U} \to \tilde{\mathscr{X}}$, then we simply write $f'(x) = \mathscr{M}(x)$, $x \in \mathscr{U}$. These observations motivate the following generalization of Definition 1.5.

Definition 1.8 (Generalized Morse–Bott properties)**.** Let $\mathscr{X}$ and $\tilde{\mathscr{X}}$ be Banach spaces over $\mathbb{K}$, and $\tilde{\mathscr{X}} \subset \mathscr{X}^*$ be a continuous embedding, and $\mathscr{U} \subset \mathscr{X}$ be an open subset, and $f : \mathscr{U} \to \mathbb{K}$ be a C^2 function such that $\operatorname{Crit} f$ is a C^2, connected submanifold and $f'(x) \in \tilde{\mathscr{X}}$ for all $x \in \mathscr{U}$.

1. If $x_0 \in \operatorname{Crit} f$ and $\operatorname{Ker} f''(x_0)$ has a closed complement and $\operatorname{Ran} f''(x_0) = \tilde{\mathscr{X}}$ and $T_{x_0} \operatorname{Crit} f = \operatorname{Ker} f''(x_0)$, then f is *Morse–Bott at the point* x_0;
2. If f is Morse–Bott at each point $x \in \operatorname{Crit} f$, then f is *Morse–Bott along* $\operatorname{Crit} f$ or a *Morse–Bott function*.

Because $f''(x_0) \in \mathscr{L}(\mathscr{X}, \mathscr{X}^*)$, then $\mathscr{K} := \operatorname{Ker} f''(x_0) \subset \mathscr{X}$ is a closed subspace, the quotient, $\mathscr{X}/\mathscr{K}$, is a Banach space, and the induced operator, $f''(x_0) \in \mathscr{L}(\mathscr{X}/\mathscr{K}, \tilde{\mathscr{X}})$, is an isomorphism by the Open Mapping Theorem. However, in our proof of the Morse–Bott Lemma (see Theorem 2.14) for functions that are Morse–Bott at a point in the sense of Definition 1.8 (1), we shall exploit the existence of a splitting, $\mathscr{X} = \mathscr{X}_0 \oplus \mathscr{K}$, where $\mathscr{X}_0 \subset \mathscr{X}$ is a closed subspace.

Again, the Morse–Bott Lemma (see Theorem 2.14) implies that if f is Morse–Bott at a point, as in Definition 1.8 (1), then f is Morse–Bott along an open neighborhood of that point in $\operatorname{Crit} f$, as in Definition 1.8 (2).

Theorem 2 (Generalized Morse–Bott property of an analytic function with Łojasiewicz exponent one half)**.** *Let $\mathscr{X}$ and $\tilde{\mathscr{X}}$ be Banach spaces over $\mathbb{K}$, and $\tilde{\mathscr{X}} \subset \mathscr{X}^*$ be a continuous embedding, and $\mathscr{U} \subset \mathscr{X}$ be an open subset, and $f : \mathscr{U} \to \mathbb{K}$ be a non-constant analytic function such that $f(0) = 0$ and $f'(0) = 0$ and $f'(x) \in \tilde{\mathscr{X}}$ for all $x \in \mathscr{U}$ and $f''(0) \in \mathscr{L}(\mathscr{X}, \tilde{\mathscr{X}})$ is a Fredholm operator with index zero. If there is a constant $C \in (0, \infty)$ such that, after possibly shrinking $\mathscr{U}$,*

$$\|f'(x)\|_{\tilde{\mathscr{X}}} \geq C|f(x)|^{1/2}, \quad \forall\, x \in \mathscr{U}, \tag{1.8}$$

then f is a Morse–Bott function in the sense of Definition 1.8.

Hence, Theorem 2 is a converse to the simpler Theorem 7 when f is *analytic* and, moreover, immediately yields Theorem 1 upon choosing $\tilde{\mathscr{X}} = \mathscr{X}^*$.

The conclusion of Theorem 2 has an interpretation similar to that of Theorem 1. Using the more general version of the Morse–Bott Lemma provided by Theorem 2.14, after possibly shrinking $\mathscr{U}$, there are an open neighborhood, $\mathscr{V} \subset \mathscr{X}$, of the origin and an analytic diffeomorphism, $\Phi : \mathscr{V} \to \mathscr{U}$, such that $\Phi(0) = 0$ and

$$f(\Phi(y)) = \frac{1}{2}\langle y, Ay\rangle, \quad \forall\, y \in \mathscr{V},$$

where $A \in \mathscr{L}_{\mathrm{sym}}(\mathscr{X}, \tilde{\mathscr{X}})$ and, letting $\mathscr{K} := \operatorname{Ker} A \subset \mathscr{X}$ denote the finite-dimensional kernel of A with closed complement, $\mathscr{X}_0 \subset \mathscr{X}$, and $\tilde{\mathscr{X}}_0 := \operatorname{Ran} A \subset \tilde{\mathscr{X}}$ denote the closed range of A with finite-dimensional complement, $\tilde{\mathscr{K}} \cong \mathscr{K} = \operatorname{Ker} A$, and $\tilde{\mathscr{X}}_0 \cong \mathscr{X}_0$ (see Lemma 2.4), we have

$$A = \begin{pmatrix} A_0 & 0 \\ 0 & 0 \end{pmatrix} : \mathscr{X}_0 \oplus \mathscr{K} \to \tilde{\mathscr{X}}_0 \oplus \tilde{\mathscr{K}},$$

where $A_0 \in \mathscr{L}(\mathscr{X}_0, \tilde{\mathscr{X}_0})$ is an isomorphism of Banach spaces that is symmetric with respect to the continuous embedding, $\tilde{\mathscr{X}_0} \subset \mathscr{X}_0^*$, and canonical pairing, $\mathscr{X}_0 \times \mathscr{X}_0^* \to \mathbb{K}$. Thus, $(f \circ \Phi)'(y) = Ay$, for all $y \in \mathscr{V}$, and $\operatorname{Crit} f \circ \Phi = \mathscr{V} \cap \operatorname{Ker} A$, an analytic submanifold of $\mathscr{V}$ of dimension equal to $\dim \operatorname{Ker} A$.

When we specialize Theorem 1 to $\mathscr{X} = \mathbb{K}^d$, we obtain the desired characterization of the optimal Łojasiewicz exponent for analytic functions on finite-dimensional vector spaces.

Corollary 3 (Morse–Bott property of an analytic function with Łojasiewicz exponent one half)**.** *Let $d \geq 1$ be an integer, $U \subset \mathbb{K}^d$ be an open neighborhood of the origin, and $f : U \to \mathbb{K}$ be a non-constant analytic function such that $f(0) = 0$ and $f'(0) = 0$. If there is a constant $C \in (0, \infty)$ such that, after possibly shrinking U, the function f obeys the Łojasiewicz gradient inequality* (1.1) *with exponent $\theta = 1/2$, then f is a Morse–Bott function.*

The interpretation of Corollary 3 is simpler than that of Theorem 1. By the Morse–Bott Lemma (Theorem 2.10 with $\mathscr{X} = \mathbb{K}^d$ and diagonalization [46, p. 278] of the symmetric matrix A over $\mathbb{K}$), after possibly shrinking U, there are an open neighborhood, $V \subset \mathbb{K}^d$, of the origin and an analytic diffeomorphism, $\Phi : V \to U$, such that $\Phi(0) = 0$ and

$$f(\Phi(y)) = \frac{1}{2} \sum_{i=1}^{d-c} a_i y_i^2, \quad \forall\, y \in V,$$

where $a_i \in \mathbb{K} \setminus \{0\}$, for $i = 1, \dots, d-c$ and an integer c obeying $0 \leq c \leq d-1$. Thus, $(f \circ \Phi)'(y) = (a_1 y_1, \dots, a_{d-c} y_{d-c}, 0, \dots, 0) \in \mathbb{K}^d$, for all $y \in V$, and $\operatorname{Crit} f \circ \Phi = \{y \in V : y_1 = \cdots = y_{d-c} = 0\}$, an analytic submanifold of V of dimension c.

1.2. **Morse Lemma for functions on Banach spaces with degenerate critical points.**

It is important to carefully distinguish between the *Morse–Bott Lemma* and the more general *Morse Lemma for functions on Banach spaces with degenerate critical points* (also known as the *Morse Lemma with parameters* or *Splitting Lemma*): the latter makes no assumption on whether the critical set is a submanifold or, even if it is a submanifold, whether its tangent space at each critical point is equal to the kernel of the Hessian operator at that point. We begin with the

Theorem 4 (Morse Lemma for functions on Banach spaces with degenerate critical points)**.** *Let $\mathscr{X}$ and $\mathscr{Y}$ be Banach spaces over $\mathbb{K}$, and $\mathscr{U} \subset \mathscr{X}$ and $\mathscr{V} \subset \mathscr{Y}$ be open neighborhoods of the origin, and $f : \mathscr{X} \times \mathscr{Y} \supset \mathscr{U} \times \mathscr{V} \ni (x, y) \mapsto f(x, y) \in \mathbb{K}$ be a C^{p+2} function ($p \geq 1$) such that $f(0, 0) = 0$ and $D_x f(0, 0) = 0$. If $D_x^2 f(0, 0) \in \mathscr{L}(\mathscr{X}, \mathscr{X}^*)$ is invertible then, after possibly shrinking $\mathscr{U}$ and $\mathscr{V}$, there are an open neighborhood $\mathscr{U}'$ of the origin and a C^p diffeomorphism, $\Phi : \mathscr{U}' \times \mathscr{V} \ni (z, y) \mapsto (x, y) \in \mathscr{U} \times \mathscr{V}$ with $\Phi(0, 0) = (0, 0)$ and*

$$D\Phi(0, 0) = \begin{pmatrix} \mathrm{id}_{\mathscr{X}} & \star \\ 0 & \mathrm{id}_{\mathscr{Y}} \end{pmatrix} \in \mathscr{L}(\mathscr{X} \oplus \mathscr{Y}) \tag{1.9}$$

such that

$$f(\Phi(z, y)) = f(\Phi(0, y)) + \frac{1}{2}\langle z, Az \rangle, \quad \forall\, (z, y) \in \mathscr{U}' \times \mathscr{V}, \tag{1.10}$$

where $A := D_x^2 f(0, 0) = D_z^2 (f \circ \Phi)(0, 0) \in \mathscr{L}_{\mathrm{sym}}(\mathscr{X}, \mathscr{X}^*)$. If f is analytic, then Φ is analytic.

Remark 1.9 (Previous versions of the Morse Lemma for functions with degenerate critical points). Theorem 4 is a generalization of [45, Lemma C.6.1] from the case where f is C^∞ and $\mathscr{H} = \mathbb{R}^d$ with its standard inner product; and [36, Lemma 1], due to Gromoll and Meyer, from the case where f is C^∞ and $\mathscr{H}$ is a real separable Hilbert space and the Hessian is Fredholm at the critical point; and [59, Theorem 1], due to Mawhin and Willem, from the case where f is C^2 and $\mathscr{H}$ is a real Hilbert space and the Hessian is Fredholm at the critical point. Hörmander's proof is similar to that of Palais [65, p. 307], who allows $\mathscr{H}$ to be a real Hilbert space but assumes that the critical point is non-degenerate. In [66], Palais uses the Moser Path Method from [62]; see [89] for another early application of [62] in the setting of Banach manifolds. Theorem 4 is also known as the Morse Lemma with parameters (see [70]) or Splitting Lemma — see Bröcker [16, Lemma 14.12] or Poston and Stewart [71, Theorems 4.5 and 6.1] — and attributed to Thom.

Remark 1.10 (Previous versions of the Morse Lemma for functions on Banach spaces). Palais proved the Morse Lemma for smooth functions on Hilbert spaces in [65, p. 307] and later extended and simplified his proof to give the Morse Lemma for smooth functions on Banach spaces in [66, p. 968] (see also Guillemin and Sternberg [37, Chapter 1, Appendix 1] for another exposition of Palais's proof in [66]).

The version of the Morse Lemma (for functions with degenerate critical points) in Theorem 4, is not the most general possible extension of Hörmander's [45, Lemma C.6.1] from Euclidean space to a Banach space. Rather, as in Lang's exposition [53, Section 7.5] the proof of Palais's version of the Morse Lemma on Hilbert spaces [65, p. 307], our hypotheses in Theorem 4 are strong enough that replacement of Euclidean space by a Banach space over $\mathbb{K}$ involves no new complication.

A shorter proof of Palais' version of the Morse Lemma on Banach spaces [66, p. 968] is given by Ang and Tuan [4]; see also their article [86]. Kuo [51, Theorem, p. 364] and Tromba [85] consider C^{p+2} functions on Banach spaces with isolated critical points obeying a more general notion of non-degeneracy than that of Palais [66, p. 968], avoiding the implication in [66] that the Banach space is isomorphic to its dual space via the isomorphism provided by the Hessian operator. A related extension was proposed earlier by Uhlenbeck [87], inspired by a question of Smale [80]. Antoine [5, Theorem 1] considers C^{p+2} functions on Banach spaces with isolated critical points and invertible Hessian operator.

Gromoll and Meyer [36, Lemma 1] consider C^∞ functions on Hilbert spaces with Fredholm Hessian operator while Mawhin and Willem [59, Theorem 1] relax the regularity requirement of Gromoll and Meyer in [36] from C^∞ to C^2.

Remark 1.11 (Regularity of the function f). It is likely that one adapt the arguments of Cambini [17], Kuiper [50], and Mawhin and Willem [59] to reduce the C^{p+2} (with $p \geq 1$) regularity requirement on f in Theorem 4 to C^2, as those authors allow for their versions of the Morse Lemma on Banach spaces, but the resulting proof would be lengthier and less elegant. See Remark 1.15 for further discussion.

Remark 1.12 (Morse Lemma for functions with degenerate critical points and Lyapunov-Schmidt reduction). Theorem 4 may be regarded as a more refined version of the technique of Lyapunov-Schmidt reduction (see Guo and Wu [38, Section 5.1], Huang [47, Proposition 2.4.1], or Nirenberg [64, Section 2.7.6]) when $\mathscr{Y}$ has finite dimension. Indeed, as we see from Lemma 3.3, Theorem 4 immediately reduces the proof of the Łojasiewicz gradient inequality for analytic functions on Banach spaces to the well-known Łojasiewicz gradient inequality for analytic functions on Euclidean space (see Feehan [31] for a detailed survey and references).

As noted in Section 1.1 (see the discussion prior to Theorem 2), the hypothesis in Theorem 4 that $D_x^2 f(0,0) \in \mathscr{L}(\mathscr{X}, \mathscr{X}^*)$ is an isomorphism is strong but is relaxed in the following generalization which immediately yields Theorem 4 upon specializing to $\tilde{\mathscr{X}} = \mathscr{X}^*$. See Kuo [51, Theorem, p. 364], Tromba [85], [87], and Mawhin and Willem [59, Theorem 1] for related refinements, though none provide the generality of Theorem 4.

Theorem 5 (Generalized Morse Lemma for functions on Banach spaces with degenerate critical points)**.** *Let $\mathscr{X}$, $\tilde{\mathscr{X}}$, and $\mathscr{Y}$ be Banach spaces over $\mathbb{K}$, and $\tilde{\mathscr{X}} \subset \mathscr{X}^*$ be a continuous embedding, and $\mathscr{U} \subset \mathscr{X}$ and $\mathscr{V} \subset \mathscr{Y}$ be open neighborhoods of the origin, and $f : \mathscr{X} \times \mathscr{Y} \supset \mathscr{U} \times \mathscr{V} \ni (x,y) \mapsto f(x,y) \in \mathbb{K}$ be a C^{p+2} function ($p \geq 1$) such that $f(0,0) = 0$ and $D_x f(0,0) = 0$ and $D_x f(x,y) \in \tilde{\mathscr{X}}$ for all $(x,y) \in \mathscr{U} \times \mathscr{V}$. If $D_x^2 f(0,0) \in \mathscr{L}(\mathscr{X}, \tilde{\mathscr{X}})$ is invertible then, after possibly shrinking $\mathscr{U}$ and $\mathscr{V}$, there are an open neighborhood of the origin, $\mathscr{U}' \subset \mathscr{X}$, and a C^p diffeomorphism, $\Phi : \mathscr{U}' \times \mathscr{V} \ni (z,y) \mapsto (x,y) = \Phi(z,y) \in \mathscr{U} \times \mathscr{V}$ with $\Phi(0,0) = (0,0)$ and*

$$D\Phi(0,0) = \begin{pmatrix} \mathrm{id}_{\mathscr{X}} & \star \\ 0 & \mathrm{id}_{\mathscr{Y}} \end{pmatrix} \in \mathscr{L}(\mathscr{X} \oplus \mathscr{Y}) \tag{1.11}$$

such that

$$f(\Phi(z,y)) = f(\Phi(0,y)) + \frac{1}{2}\langle z, Az\rangle, \quad \forall\, (z,y) \in \mathscr{U}' \times \mathscr{V}, \tag{1.12}$$

where $A := D_x^2 f(0,0) = D_z^2 (f \circ \Phi)(0,0) \in \mathscr{L}_{\mathrm{sym}}(\mathscr{X}, \tilde{\mathscr{X}})$. If f is analytic, then Φ is analytic.

1.3. **Morse and Morse–Bott Lemmas for functions on Banach spaces.** Theorems 4 and 5 easily yield versions of the Morse Lemma and, more broadly, the Morse–Bott Lemma in varying degrees of generality, namely Theorems 2.8, 2.10, 2.14, and 2.15: We refer to Section 2.4 for their statements and short proofs.

1.4. **Łojasiewicz–Simon gradient inequalities for smooth Morse–Bott functions on Banach spaces.** The Morse–Bott Lemma (Theorems 2.10 and 2.14) readily leads to Łojasiewicz–Simon gradient inequalities with exponent one half for C^{p+2} (with $p \geq 1$) Morse–Bott functions on Banach spaces, giving alternative proofs to those that do not on the Morse–Bott Lemma provided by the author in [31, Theorem 3] (when f is C^2 and $\mathscr{X}$ is finite-dimensional) and [30, Theorem 3 and Corollaries 4 and 5] (when f is C^2 and $\mathscr{X}$ is a Banach space) and by the author and Maridakis [32, Theorem 4] (when f is C^2 and $\mathscr{X}$ is a Banach space and $f''(0)$ has certain Fredholm properties). We begin with the following analogue of Theorem 1 and which is similar to [30, Corollary 5], which is proved with appealing to the Morse–Bott Lemma provided by Theorem 2.10, except that here we assume that f is C^{p+2} for some $p \geq 1$.

Theorem 6 (Łojasiewicz gradient inequality for C^{p+2} Morse–Bott functions on Banach spaces)**.** *(Compare* [30, Corollary 5]*.) Let $\mathscr{X}$ be a Banach space over $\mathbb{K}$, and $f : \mathscr{U} \to \mathbb{K}$ be a C^{p+2} function ($p \geq 1$) such that $f(0) = 0$ and $f'(0) = 0$. If f is Morse–Bott at the origin in the sense of Definition 1.5 (1) then, after possibly shrinking $\mathscr{U}$, there is a constant $C \in (0, \infty)$ such that*

$$\|f'(x)\|_{\mathscr{X}^*} \geq C|f(x)|^{1/2}, \quad \forall\, x \in \mathscr{U}. \tag{1.13}$$

Remark 1.13 (On the definition of a Morse–Bott function on a Banach space). In [30, Definition 1.5] and [32, Definition 1.9], we say that a C^2 function, $f : \mathscr{U} \to \mathbb{K}$, is Morse–Bott at a point $x_0 \in \mathscr{U}$ if $\operatorname{Crit} f$ is a C^2 (connected) submanifold and $T_{x_0} \operatorname{Crit} f = \operatorname{Ker} f''(x_0)$, but omit requirements that $\operatorname{Ker} f''(x_0) \subset \mathscr{X}$ have a closed complement $\mathscr{X}_0 \subset \mathscr{X}$ or that $\operatorname{Ran} f''(x_0) = \mathscr{X}_0^*$. In our Łojasiewicz gradient inequality [30, Corollary 5] for C^2 Morse–Bott functions analogous to Theorem 6, we required that $\operatorname{Ker} f''(x_0)$ have a closed complement $\mathscr{X}_0$ but only that $\operatorname{Ran} f''(x_0) \subset \mathscr{X}^*$ be a closed subspace (equivalent to $\operatorname{Ran} f''(x_0) = \mathscr{X}_0^*$ when $\mathscr{X}$ is reflexive by Lemma 2.3 (2). In the hypotheses for our [32, Theorem 4], we impose the stronger requirement that $f''(x_0) \in \mathscr{L}(\mathscr{X}, \mathscr{X}^*)$ be Fredholm, so those additional conditions are automatic — see Lemma 2.2 (2).

We have the following analogue of Theorem 2, which is like [30, Corollary 4] (proved without appealing to the Morse–Bott Lemma provided by Theorem 2.14), except that here we assume that f is C^{p+2} for some $p \geq 1$.

Theorem 7 (Generalized Łojasiewicz gradient inequality for C^{p+2} Morse–Bott functions on Banach spaces)**.** *(Compare* [30, Corollary 4]*.) Let $\mathscr{X}$ and $\tilde{\mathscr{X}}$ be Banach spaces over $\mathbb{K}$, and $\tilde{\mathscr{X}} \subset \mathscr{X}^*$ be a continuous embedding, and $f : \mathscr{U} \to \mathbb{K}$ be a C^{p+2} function ($p \geq 1$) such that $f(0) = 0$ and $f'(0) = 0$ and $f'(x) \in \tilde{\mathscr{X}}$ for all $x \in \mathscr{U}$. If f is Morse–Bott at the origin in the sense of Definition 1.8 (1) then, after possibly shrinking $\mathscr{U}$, there is a constant $C \in (0, \infty)$ such that*

$$\|f'(x)\|_{\tilde{\mathscr{X}}} \geq C|f(x)|^{1/2}, \quad \forall\, x \in \mathscr{U}. \tag{1.14}$$

Remark 1.14 (On the definition of a generalized Morse–Bott function on a Banach space)*.* In [30, Definition 1.5] and [32, Definition 1.9], we say that a C^2 function, $f : \mathscr{U} \to \mathbb{K}$, is Morse–Bott at a point $x_0 \in \mathscr{U}$ if $\operatorname{Crit} f$ is a C^2 (connected) submanifold and $T_{x_0} \operatorname{Crit} f = \operatorname{Ker} f''(x_0)$, but omit requirements that $\operatorname{Ker} f''(x_0) \subset \mathscr{X}$ have a closed complement $\mathscr{X}_0 \subset \mathscr{X}$ or that $\operatorname{Ran} f''(x_0) = \tilde{\mathscr{X}}$. In the hypotheses for our Łojasiewicz gradient inequality [30, Corollary 4] for C^2 Morse–Bott functions analogous to Theorem 7, we required that $\operatorname{Ker} f''(x_0) \subset \mathscr{X}$ have a closed complement in $\mathscr{X}$ but only that $\operatorname{Ran} f''(x_0) \subset \tilde{\mathscr{X}}$ be a closed subspace. In the hypotheses for our [32, Theorem 4], we impose the stronger requirement that $f''(x_0) \in \mathscr{L}(\mathscr{X}, \tilde{\mathscr{X}})$ be Fredholm, so those additional conditions are automatic — see Lemma 2.4 (2).

Theorem 7 immediately yields Theorem 6 upon choosing $\tilde{\mathscr{X}} = \mathscr{X}^*$.

Remark 1.15 (On proofs of the Łojasiewicz–Simon gradient inequalities via Morse–Bott Lemmas)*.* As we shall see in Section 3.1, the Łojasiewicz–Simon gradient inequalities with exponent one half for C^{p+2} Morse–Bott functions ($p \geq 1$) on Banach spaces are indeed easy consequences of the Morse–Bott Lemma (Theorems 2.10 and 2.14). However, the most useful version of such a Łojasiewicz–Simon gradient inequality (namely [30, Theorem 3]), matching the generality of Theorem 10, does not appear to be an obvious consequence of a Morse–Bott Lemma. Second, because our primary focus in this article is on the Morse–Bott property of analytic functions with Łojasiewicz exponent one half, we have not striven to reduce the regularity requirements on f from C^{p+2} (with $p \geq 1$) to C^2. Mawhin and Willem [59, Theorem 1] do establish a Morse–Bott Lemma for functions that are only C^2, but impose additional hypotheses on f that we do not in our Theorem 2.14, namely that $\mathscr{X}$ be a Hilbert space and (after identifying $\mathscr{X}^* = \mathscr{X}$) that $f''(0) \in \mathscr{L}(\mathscr{X})$ be Fredholm; Mawhin and Willem generalize earlier Morse–Bott Lemmas for functions that are only C^2 due to Cambini [17], Hofer [43, 44], and Kuiper [50].

Remark 1.16 (Relationship between the Morse–Bott and other integrability conditions)*.* Given an open neighborhood $\mathscr{U}$ of a point x_0 in a Banach space $\mathscr{X}$ over $\mathbb{K}$, the property that a C^2 function $f : \mathscr{U} \to \mathbb{K}$ be Morse–Bott at x_0 (as in Definition 1.8 (1)), is closely related to the integrability condition[5] ($\star$) described by Adams and Simon in [2, pp. 229–230] and inspired by an earlier definition due to Allard and Almgren [3]: According to [2], a critical point x_0 is *integrable* if

$$(\star) \quad \forall\, v \in K,\ \exists\, u \in C^0((0,1); \tilde{\mathscr{X}}) \text{ such that } O(u) \subset \operatorname{Crit} \mathscr{E}$$
$$\text{and } \lim_{t \downarrow 0} u(t) = 0 \text{ (in } \tilde{\mathscr{X}}) \text{ and } \lim_{t \downarrow 0} u(t)/t = v \text{ (in } \tilde{\mathscr{G}}),$$

[5]I am grateful to Otis Chodosh for reminding of this condition.

where $O(u) := \{u(t) : t \in (0,1)\}$ and $\tilde{\mathscr{G}}$ is a Banach space with continuous embeddings, $\tilde{\mathscr{X}} \subset \tilde{\mathscr{G}} \subset \mathscr{X}^*$, as in the hypotheses of Theorem 10. (Adams and Simon choose $\tilde{\mathscr{G}}$ to be a certain Hilbert space but do not otherwise precisely specify the regularity properties of the path u in their definition.)

At first glance, the Adams–Simon integrability condition is weaker than Definition 1.8 (1) since in the latter definition we assume that $\operatorname{Crit} f$ is a C^2 submanifold with $T_{x_0} \operatorname{Crit} f = \operatorname{Ker} f''(x_0)$. It is likely that in the proof of our result [32, Theorem 4] with Maridakis, the Adams–Simon integrability condition could replace the property that f be Morse–Bott at a point. We refer to Appendix C for a discussion of integrability and Morse–Bott conditions for the harmonic map energy function, together with examples.

1.5. **Łojasiewicz–Simon gradient inequalities for analytic functions on Banach spaces.** Theorems 4 and 5 can be used to give alternative proofs of the Łojasiewicz–Simon gradient inequalities for analytic functions on Banach spaces which complement those of the author and Maridakis in [32] and weaken[6] their hypotheses.

Theorem 8 (Łojasiewicz–Simon gradient inequality for analytic functions on Banach spaces)**.** *(Compare* [32, Theorem 1]*.) Let $\mathscr{X}$ be a Banach space over $\mathbb{K}$, and $\mathscr{U} \subset \mathscr{X}$ be an open neighborhood of the origin, and $f : \mathscr{U} \to \mathbb{K}$ be an analytic function such that $f(0) = 0$ and $f'(0) = 0$. If $f''(0) \in \mathscr{L}(\mathscr{X}, \mathscr{X}^*)$ is a Fredholm operator with index zero then, after possibly shrinking $\mathscr{U}$, there are constants $C \in (0,\infty)$ and $\theta \in [1/2, 1)$ such that*

$$\|f'(x)\|_{\mathscr{X}^*} \geq C|f(x)|^\theta, \quad \forall\, x \in \mathscr{U}. \tag{1.15}$$

The following generalization of Theorem 8 relaxes the strong hypothesis that $f''(0) \in \mathscr{L}(\mathscr{X}, \mathscr{X}^*)$ be Fredholm and immediately yields Theorem 8 upon specializing to $\tilde{\mathscr{X}} = \mathscr{X}^*$.

Theorem 9 (Generalized Łojasiewicz–Simon gradient inequality for analytic functions on Banach spaces)**.** *(Compare* [32, Theorem 2]*.) Let $\mathscr{X}$ and $\tilde{\mathscr{X}}$ be Banach spaces over $\mathbb{K}$ with continuous embedding, $\tilde{\mathscr{X}} \subset \mathscr{X}^*$, and $\mathscr{U} \subset \mathscr{X}$ be an open neighborhood of the origin, $f : \mathscr{U} \to \mathbb{K}$ be an analytic function with $f(0) = 0$ and $f'(0) = 0$ and $f'(x) \in \tilde{\mathscr{X}}$ for all $x \in \mathscr{U}$. If $f''(0) \in \mathscr{L}(\mathscr{X}, \tilde{\mathscr{X}})$ is Fredholm with index zero then, after possibly shrinking $\mathscr{U}$, there are constants $C \in (0,\infty)$ and $\theta \in [1/2, 1)$ such that*

$$\|f'(x)\|_{\tilde{\mathscr{X}}} \geq C|f(x)|^\theta, \quad \forall\, x \in \mathscr{U}. \tag{1.16}$$

Theorem 9 is deduced from Theorem 5 in Section 3. While Theorem 9 is sufficient for many applications in geometric analysis, it also excludes some important examples (see [32, Section 1.2] for a discussion of such examples), including Simon's [76, Theorem 3]. We call a bilinear form, $b : \mathscr{X} \times \mathscr{X} \to \mathbb{K}$, *definite* if $b(x,x) \neq 0$ for all $x \in \mathscr{X} \setminus \{0\}$. We say that a continuous *embedding* of a Banach space into its continuous dual space, $\jmath : \mathscr{X} \to \mathscr{X}^*$, is *definite* if the pullback of the canonical pairing, $\mathscr{X} \times \mathscr{X} \ni (x,y) \mapsto \langle x, \jmath(y)\rangle_{\mathscr{X} \times \mathscr{X}^*} \in \mathbb{K}$, is a definite bilinear form. The following generalization of Theorem 9 does not appear to be a simple consequence of a Morse Lemma for degenerate critical points like Theorem 5.

Theorem 10 (Generalized Łojasiewicz–Simon gradient inequality for analytic functions on Banach spaces)**.** *(See* [32, Theorem 3]*.) Let $\mathscr{X}$ and $\tilde{\mathscr{X}}$ be Banach spaces over $\mathbb{R}$ with continuous*

[6]By relaxing the hypotheses that the continuous embedding, $\mathscr{X} \subset \mathscr{X}^*$, be definite, that one has a continuous embedding, $\mathscr{X} \subset \tilde{\mathscr{X}}$, and that $\mathbb{K} = \mathbb{R}$.

embeddings, $\mathscr{X} \subset \tilde{\mathscr{X}} \subset \mathscr{X}^*$, and such that the embedding, $\mathscr{X} \subset \mathscr{X}^*$, is definite. Let $\mathscr{U} \subset \mathscr{X}$ be an open subset and $f : \mathscr{U} \to \mathbb{R}$ be an analytic function such that $f(0) = 0$ and $f'(0) = 0$. Let

$$\mathscr{X} \subset \mathscr{G} \subset \tilde{\mathscr{G}} \quad \text{and} \quad \tilde{\mathscr{X}} \subset \tilde{\mathscr{G}} \subset \mathscr{X}^*,$$

be continuous embeddings of Banach spaces such that the compositions,

$$\mathscr{X} \subset \mathscr{G} \subset \tilde{\mathscr{G}} \quad \text{and} \quad \mathscr{X} \subset \tilde{\mathscr{X}} \subset \tilde{\mathscr{G}},$$

induce the same embedding, $\mathscr{X} \subset \tilde{\mathscr{G}}$. Let $\mathscr{M} : \mathscr{U} \to \tilde{\mathscr{X}}$ be a gradient map for f in the sense of Definition 1.7. Suppose that for each $x \in \mathscr{U}$, the bounded, linear operator,

$$\mathscr{M}'(x) : \mathscr{X} \to \tilde{\mathscr{X}},$$

has an extension

$$\mathscr{M}_1(x) : \mathscr{G} \to \tilde{\mathscr{G}}$$

such that the map

$$\mathscr{U} \ni x \mapsto \mathscr{M}_1(x) \in \mathscr{L}(\mathscr{G}, \tilde{\mathscr{G}}) \quad \text{is continuous.}$$

If $\mathscr{M}'(x_\infty) : \mathscr{X} \to \tilde{\mathscr{X}}$ and $\mathscr{M}_1(x_\infty) : \mathscr{G} \to \tilde{\mathscr{G}}$ are Fredholm operators with index zero then, after possibly shrinking $\mathscr{U}$, there are constants $C \in (0, \infty)$ and $\theta \in [1/2, 1)$ such that

$$\|\mathscr{M}(x)\|_{\tilde{\mathscr{G}}} \geq C|f(x)|^\theta, \quad \forall\, x \in \mathscr{U}. \tag{1.17}$$

Suppose now that $\tilde{\mathscr{G}} = \mathscr{H}$, a Hilbert space, so that the embedding $\mathscr{G} \subset \mathscr{H}$ in Theorem 10, factors through $\mathscr{G} \subset \mathscr{H} \simeq \mathscr{H}^*$ and therefore

$$f'(x)v = \langle v, \mathscr{M}(x)\rangle_{\mathscr{X} \times \mathscr{X}^*} = (v, \mathscr{M}(x))_{\mathscr{H}}, \quad \forall\, x \in \mathscr{U} \text{ and } v \in \mathscr{X},$$

using the continuous embeddings, $\tilde{\mathscr{X}} \subset \mathscr{H} \subset \mathscr{X}^*$. As we note in Remark 1.17, the hypothesis in Theorem 10 that the embedding, $\mathscr{X} \subset \mathscr{X}^*$, is definite is implied by the assumption that $\mathscr{X} \subset \mathscr{H}$ is a continuous embedding into a Hilbert space. Theorem 10 then yields

$$\|\mathscr{M}(x)\|_{\mathscr{H}} \geq C|f(x)|^\theta, \tag{1.18}$$

as desired.

Remark 1.17 (Comments on the embedding hypothesis in Theorem 10). (See [32, Remark 1.1].) The hypothesis in Theorem 10 on the continuous embedding, $\mathscr{X} \subset \mathscr{X}^*$, is easily achieved given a continuous embedding ε of $\mathscr{X}$ into a Hilbert space $\mathscr{H}$. Indeed, because $\langle y, \jmath(x)\rangle_{\mathscr{X} \times \mathscr{X}^*} = (\varepsilon(y), \varepsilon(x))_{\mathscr{H}}$ for all $x, y \in \mathscr{X}$, then $\langle x, \jmath(x)\rangle_{\mathscr{X} \times \mathscr{X}^*} = 0$ implies $x = 0$; see [15, Remark 3, page 136] or [33, Lemma D.1] for details.

1.6. **Applications.** Due to the difficulty in computing the Łojasiewicz exponent, one should not in general expect Theorems 1 or 2 to provide a useful way to prove the Morse–Bott property of an analytic function with Hessian operator that is Fredholm of index zero. Nonetheless, they provide insight to applications in geometric analysis and we survey a few such applications here.

1.6.1. *Yamabe energy function, integrability conditions, Łojasiewicz exponents, and Morse–Bott properties.* Carlotto, Chodosh, and Rubinstein [19] study the existence of 'slowly-converging' (volume-normalized) gradient flows for the Yamabe energy function on Riemannian metrics over a closed manifold of dimension three or more with the aid of results due to *a*) Adams and Simon [2] on the relationship between integrability and certain types of non-integrability and rates of convergence of geometric flows, and *b*) Chill [20] on the Łojasiewicz–Simon gradient inequality for functions on Banach spaces. In particular, for a certain class of geometric flows, Adams and Simon show that the integrability condition ($\star$) implies an exponential rate of convergence [2, Theorem 1 (i)] and in a certain subcase where integrability fails [2, Theorem 1 (ii)], the flow converges according to a negative power law and thus is slowly converging in the terminology of [19]. We refer the reader to Appendix A for an exposition of our general result [29, Theorem 3] on the relationship between the rate of convergence for the gradient flow of a function obeying a Łojasiewicz–Simon gradient inequality near a critical point and the value of the Łojasiewicz exponent.

When $\mathscr{E}$ is the Yamabe (or Einstein–Hilbert) energy function, Carlotto, Chodosh, and Rubinstein show that the Adams–Simon integrability condition ($\star$) implies that the Łojasiewicz exponent for $\mathscr{E}$ at a critical point is equal to one half [19, Proposition 13], for a suitable choice of Banach spaces, and that in turn indicates (by the main results of this article) that $\mathscr{E}$ should be Morse–Bott at the critical point. More generally, when $\mathscr{E}$ is an analytic function on a Banach space obeying the hypotheses similar to those of Theorems 1, 2, or perhaps even Theorem 10, we would expect the Adams–Simon integrability condition ($\star$) for a critical point to imply that $\mathscr{E}$ is Morse–Bott at that point by generalizing the proof due to Kwon [52] of her Proposition C.2.

1.6.2. *Harmonic map energy function for maps from a Riemann surface into a closed Riemannian manifold.* For background on harmonic maps, we refer to Hélein [40], Jost [48], Simon [78], Struwe [81], and references cited therein. Let (M,g) and (N,h) be a pair of closed, smooth Riemannian manifolds. One defines the *harmonic map energy function* by

$$\mathscr{E}_{g,h}(f) := \frac{1}{2}\int_M |df|_{g,h}^2\, d\operatorname{vol}_g, \tag{1.19}$$

for smooth maps, $f : M \to N$, where $df : TM \to TN$ is the differential map

For the harmonic map energy function, a Łojasiewicz gradient inequality with exponent one half,

$$\|\mathscr{E}'(f)\|_{L^p(S^2)} \geq Z|\mathscr{E}(f) - \mathscr{E}(f_\infty)|^{1/2},$$

has been obtained by Kwon [52, Theorem 4.2] for maps $f : S^2 \to N$, where N is a closed Riemannian manifold and f is close to a harmonic map f_∞ in the sense that

$$\|f - f_\infty\|_{W^{2,p}(S^2)} < \sigma,$$

where p is restricted to the range $1 < p \leq 2$, and f_∞ is assumed to be *integrable* in the sense of [52, Definitions 4.3 or 4.4 and Proposition 4.1]. Her proof of [52, Proposition 4.1] quotes results of Simon [77, pp. 270–272] and Adams and Simon [2, Lemma 1]. The result [57, Lemma 3.3] due to Liu and Yang is another example of Łojasiewicz gradient inequality with exponent one half for the harmonic map energy function, but restricted to the setting of maps $f : S^2 \to N$, where N is a Kähler manifold of complex dimension $n \geq 1$ and nonnegative bisectional curvature, and the energy $\mathscr{E}(f)$ is sufficiently small. The result of Liu and Yang generalizes that of Topping [84, Lemma 1], who assumes that $N = S^2$.

Milnor observes [60, Footnote to Problem 3-c] that the space of holomorphic maps of degree d from $\mathbb{CP}^1$ to $\mathbb{CP}^1$ is a non-compact complex manifold of dimension $2d+1$. However, he notes [60, Footnote to Problem 3-c] that there is an example (due to J. Harris) of a Riemann surface Σ of genus 5 such that the space of holomorphic maps from Σ into $\mathbb{CP}^1$ has singularities. In general, the space of harmonic maps of degree d from S^2 into S^{2n} (with $n \geq 1$) will not be a smooth manifold [34]. We survey some positive results for spaces of harmonic maps in Appendix C. The version of the 'Bumpy Metric Theorem' proved by Moore as [61, Theorem 5.1.1] states that if M is a compact manifold of dimension at least three and the Riemannian metric is generic, then all minimal two-spheres in M are as nondegenerate as allowed by the group of conformal automorphisms of S^2, that is, they lie on nondegenerate critical submanifolds of $\operatorname{Map}(S^2, M)$, each such submanifold being an orbit for the symmetry group $\mathrm{PSL}(2,\mathbb{C})$.

1.6.3. *Moduli spaces of flat connections and representation varieties.* When a base manifold X is compact and Kähler[7], and G is a complex reductive Lie group, Simpson proved that the singularities in the moduli space of flat connections are at worst quadratic at any reductive representation of the fundamental group [79, Corollary 2.4]; when G is a compact Lie group, this result is due to Goldman and Millson [35, Theorem 1]. When the base manifold X is not compact or Kähler then the singularities in the moduli space of flat connections may be worse. Indeed, this can occur for representation varieties for fundamental groups of certain closed, smooth three-dimensional manifolds. Goldman and Millson [35, Section 9.1] choose $X = H/\Gamma$, where H is the three-dimensional real Heisenberg group and $\Gamma \subset H$ is a lattice, so that X is the total space of an oriented circle bundle over a two-torus with non-zero Euler class. If G is an algebraic Lie group that is not two-step nilpotent and $\rho : \Gamma \to G$ is the trivial representation, then the representation variety, $\mathscr{R}(\Gamma, G)$, is not quadratic at ρ: the analytic germ of $\mathscr{R}(\Gamma, G)$ is isomorphic to a cubic cone. A compact Lie group with a simple Lie algebra, such as $\mathrm{SU}(n)$ for $n \geq 2$ is not nilpotent and so we may choose $G = \mathrm{SU}(n)$ with $n \geq 2$ in the Goldman–Millson counterexample. Recall [26, Proposition 2.2.3] that the gauge-equivalence classes of flat connections on a principal G-bundle over a connected manifold, X, are in one-to-one correspondence with the conjugacy classes of representations, $\pi_1(X) \to G$.

At first glance, there might appear to be a contradiction between the Goldman–Millson counterexample [35, Section 9.1] and our [30, Theorem 2], which asserts that the Yang–Mills energy function,

$$\mathscr{E}(A) := \frac{1}{2}\int_X |F_A|^2 \, d\,\mathrm{vol}_g, \tag{1.20}$$

on the affine space of $W^{1,q}$ connections A on a principal G-bundle (for a compact Lie group G) over a closed Riemannian manifold (X, g) of dimension $d \geq 2$ (and $q \in [2, \infty)$ obeying $q > d/2$) satisfies the following Łojasiewicz gradient inequality with exponent one half,

$$\|\mathscr{E}'(A)\|_{W^{-1,2}(X;\Lambda^1\otimes\mathrm{ad}P)} \geq C\, \mathscr{E}(A)^{1/2}, \tag{1.21}$$

provided the curvature F_A obeys

$$\|F_A\|_{L^{s_0}(X;\Lambda^2\otimes\mathrm{ad}P)} \leq \varepsilon, \tag{1.22}$$

where $s_0 = d/2$ when $d \geq 3$ or $s_0 > 1$ when $d = 2$. However, as noted in [30, Remark 1.4], the inequality (1.21) may be proved in two ways. One approach (see [30, Theorem 7]) is to assume that A is norm-close to a *regular* flat connection Γ on P and then apply our general Łojasiewicz gradient inequality for a C^2 Morse–Bott function on an abstract Banach space [30, Theorem 3]. The approach we take in our proof of [30, Theorem 2] is to instead use (1.22) to infer the existence of a flat connection Γ that is norm-close to A via our [30, Theorem 1] (which generalizes a result due to Uhlenbeck [88]). However, the construction of Γ in our proof of [30, Theorem 1] suggests that $[\Gamma]$ is a generic point in the moduli space of flat connections on P (a compact real analytic space) and thus Γ should be a regular point. In particular, [30, Theorem 2] does *not* assert that the Yang–Mills energy function obeys a Łojasiewicz gradient inequality with exponent one half at *any* flat connection or imply that all flat connections are regular points.

[7]I am grateful to Graeme Wilkin for drawing my attention to the results of Simpson and Goldman–Millson described here.

1.6.4. *F-function on the space of hypersurfaces in Euclidean space.* Colding and Minicozzi [21, 22] have given proofs of Łojasiewicz–Simon gradient and distance inequalities [23, Equations (5.9) and (5.10)] that do not involve Lyapunov-Schmidt reduction to a finite-dimensional gradient inequality, as in the original paradigm due to Simon [76]. Their gradient inequality applies to the F function [23, Section 2.4] on the space of hypersurfaces $\Sigma \subset \mathbb{R}^{d+1}$ and is analogous to (1.1) with $\theta = 2/3$. Their cited articles contain detailed technical statements of their inequalities while their article with Pedersen [23] contains a less technical summary of some of their main results.

1.7. **Structure of this article.** In Section 2, we prove the Morse Lemma for functions on Banach spaces with degenerate critical points (Theorem 5) and then deduce some corollaries, including the Morse Lemma for functions on Banach spaces with non-degenerate critical points (Theorem 2.8), and the Morse–Bott Lemma for functions on Banach spaces (Theorems 2.10 and 2.14). In Section 3, we apply Theorem 2.14 to prove the Łojasiewicz gradient inequality for C^{p+2} Morse–Bott functions on Banach spaces (Theorem 7) and apply Theorem 5 to prove the Łojasiewicz gradient inequality for analytic functions on Banach spaces (Theorem 9). Finally, in Section 4 we complete the proof of the Morse–Bott property of an analytic function with Łojasiewicz exponent one half (Theorem 1). In Appendix A, we discuss our general result [29, Theorem 3] on the rate of convergence of a gradient flow for a function obeying a Łojasiewicz gradient inequality. In Appendix B, we describe the relationship between Morse–Bott functions and quadratic simple normal crossing functions. In Appendix C, we survey results on integrability conditions and the Morse-Bott property for critical points of the harmonic map energy function.

1.8. **Acknowledgments.** I am indebted to Michael Greenblatt and András Némethi for independently pointing out to me that, for functions on Euclidean space, the Morse Lemma for functions with degenerate critical points (also known as the Morse Lemma with parameters or Splitting Lemma) should be the key ingredient needed to prove the main result of this article in the finite-dimensional case (Corollary 3). I also thank Carles Bivià-Ausina, Otis Chodosh, Tristan Collins, Santiago Encinas, Luis Fernandez, Antonella Grassi, David Hurtubise, Johan de Jong, Qingyue Liu, Doug Moore, Yanir Rubinstein, Siddhartha Sahi, Ovidiu Savin, Peter Topping, Graeme Wilkin, and Jarek Włodarczyk for helpful communications, discussions, or questions during the preparation of this article. I am grateful to the National Science Foundation for their support and the Dublin Institute for Advanced Studies and Yi-Jen Lee and the Institute of Mathematical Sciences at the Chinese University of Hong Kong for their hospitality and support during the preparation of this article.

2. Generalized Morse Lemmas for functions on Banach spaces

In Sections 2.1 and 2.2, respectively, we collect some basic observations from linear and nonlinear functional analysis that we require in this article. In Section 2.3, we prove the Morse Lemma for functions on Banach spaces with degenerate critical points (Theorem 5) and in Section 2.4 we deduce some corollaries, including the Morse Lemma for functions on Banach spaces with non-degenerate critical points (Theorem 2.8), and the Morse–Bott Lemma for functions on Banach spaces (Theorems 2.10 and 2.14).

2.1. **Preliminaries on linear functional analysis.** In this subsection, we gather a few elementary observations from linear functional analysis. We begin with the following useful

Lemma 2.1 (Dual space of a direct sum of Banach spaces)**.** *(See* [28]*.) If $\mathscr{X}, \mathscr{Y}$ are Banach spaces over $\mathbb{K}$, then $(\mathscr{X} \oplus \mathscr{Y})^* = \mathscr{X}^* \oplus \mathscr{Y}^*$.*

Proof. Let $\mathscr{Z} := \mathscr{X} \oplus \mathscr{Y}$, with product norm $\|(x,y)\|_{\mathscr{X}\oplus\mathscr{Y}} := \|x\|_{\mathscr{X}} + \|y\|_{\mathscr{Y}}$, continuous projection operators, $\pi_{\mathscr{X}} : \mathscr{Z} \to \mathscr{X}$ and $\pi_{\mathscr{Y}} : \mathscr{Z} \to \mathscr{Y}$, continuous injection operators, $\iota_{\mathscr{X}} : \mathscr{X} \to \mathscr{Z}$ and $\iota_{\mathscr{Y}} : \mathscr{Y} \to \mathscr{Z}$, and define $T : \mathscr{Z}^* \to \mathscr{X}^* \oplus \mathscr{Y}^*$ by $Tz^* := (z^*\iota_{\mathscr{X}}, z^*\iota_{\mathscr{Y}})$. We observe that T is bounded because

$$\|Tz^*\|_{\mathscr{X}^*\oplus\mathscr{Y}^*} = \|z^*\iota_{\mathscr{X}}\|_{\mathscr{X}^*} + \|z^*\iota_{\mathscr{Y}}\|_{\mathscr{Y}^*} \le 2\|z^*\|_{\mathscr{Z}^*},$$

noting that

$$\|z^*\iota_{\mathscr{X}}\|_{\mathscr{X}^*} = \sup_{x\in\mathscr{X}\setminus\{0\}} \frac{|z^*(\iota_{\mathscr{X}}(x))|}{\|x\|_{\mathscr{X}}} \le \sup_{z\in\mathscr{Z}\setminus\{0\}} \frac{|z^*(z)|}{\|z\|_{\mathscr{Z}}} = \|z^*\|_{\mathscr{Z}^*},$$

and similarly $\|z^*\pi_{\mathscr{Y}}\|_{\mathscr{X}^*} \le \|z^*\|_{\mathscr{Z}^*}$. The operator T is injective since if $Tz^* = 0$, then $(z^*(\iota_{\mathscr{X}}(x)), z^*(\iota_{\mathscr{Y}}(y))) = 0$ for all $(x,y) \in \mathscr{X} \oplus \mathscr{Y}$ and so $z^*(z) = 0$ for all $z = x + y \in \mathscr{Z}$ and hence $z^* = 0$. The operator T is surjective since if $x^* \in \mathscr{X}^*$ and $y^* \in \mathscr{Y}^*$ and we define $z^*(z) := x^*(x) + y^*(y)$ for $z = x + y \in \mathscr{Z}$, then $z^* \in \mathscr{Z}^*$. The operator T^{-1} is bounded by the Open Mapping Theorem or by observing that $T^{-1}(x^*, y^*) = x^*\pi_{\mathscr{X}} + y^*\pi_{\mathscr{Y}} \in \mathscr{Z}^*$. Therefore, T is an isomorphism of Banach spaces. □

In the proof of Lemma 2.1, we note that the adjoint, $\iota_{\mathscr{X}}* : \mathscr{Z}^* \to \mathscr{X}^*$, is continuous and $(\iota_{\mathscr{X}}^* z^*)(x) = z^*(\iota_{\mathscr{X}}(x))$ for all $x \in \mathscr{X}$, so if $x^* \in \mathscr{X}^*$, then $(\iota_{\mathscr{X}}^* x^*)(x) = x^*(\iota_{\mathscr{X}}(x)) = x^*$ for all $x \in \mathscr{X}$. Thus, $\pi_{\mathscr{X}^*} = \iota_{\mathscr{X}}^* : \mathscr{Z}^* \to \mathscr{X}^*$ and $\pi_{\mathscr{Y}^*} = \iota_{\mathscr{Y}}^* : \mathscr{Z}^* \to \mathscr{Y}^*$ are the induced projection operators. The following lemma helps motivate Definition 1.5 (1) but is not used elsewhere in this article.

Lemma 2.2 (Range of a symmetric operator whose kernel has closed complement)**.** *Let $\mathscr{X}$ be a Banach space over $\mathbb{K}$. If $A \in \mathscr{L}(\mathscr{X}, \mathscr{X}^*)$ is symmetric and* $\operatorname{Ker} A$ *has closed complement, $\mathscr{X}_0$, in $\mathscr{X}$ then the following hold:*

(1) $\operatorname{Ran} A \subset \mathscr{X}_0^*$*;*
(2) If A is Fredholm with index zero, then $\operatorname{Ran} A = \mathscr{X}_0^*$*.*

Proof. By hypothesis on $\mathscr{K} := \operatorname{Ker} A$, we have $\mathscr{X} = \mathscr{X}_0 \oplus \mathscr{K}$ and thus $\mathscr{X}^* = \mathscr{X}_0^* \oplus \mathscr{K}^*$ by Lemma 2.1. Suppose $\alpha \in \mathscr{K}^* \cap \operatorname{Ran} A$, so $\alpha = Ax$ for some $x \in \mathscr{X}$. If $\xi \in \mathscr{K}$, then $\langle \xi, \alpha\rangle = \langle \xi, Ax\rangle = \langle x, A\xi\rangle$, since A is symmetric, and $\langle x, A\xi\rangle = 0$ as $\xi \in \operatorname{Ker} A$. Because $\xi \in \mathscr{K}$ was arbitrary, we see that $\alpha = 0$ on $\mathscr{K}^*$ and as $\alpha = 0$ on $\mathscr{X}_0$ (because $\mathscr{X}^* = \mathscr{X}_0^* \oplus \mathscr{K}^*$ and $\alpha \in \mathscr{K}^*$), we obtain $\alpha = 0$ on $\mathscr{X}$ and $\alpha = 0 \in \mathscr{X}^*$. Hence, $\operatorname{Ran} A \subset \mathscr{X}_0^*$, as claimed in Item (1).

If A is Fredholm, then $\mathscr{K}$ is finite-dimensional and thus has a closed complement, $\mathscr{X}_0$, by [73, Lemma 4.21 (a)]. Item (1) implies that $\operatorname{Ran} A \subset \mathscr{X}_0^*$. Because A has index zero, then $\dim \operatorname{Ker} A = \dim(\mathscr{X}^*/\operatorname{Ran} A)$ and because $\mathscr{X}^*/\operatorname{Ran} A$ is finite-dimensional, $\operatorname{Ran} A$ has a closed complement, say $\mathscr{M}$, with $\mathscr{X}^* = \operatorname{Ran} A \oplus \mathscr{M}$, by [73, Lemma 4.21 (b)] and $\mathscr{X}^*/\operatorname{Ran} A = \mathscr{M}$. But $\dim \mathscr{K}^* = \dim \mathscr{K} = \dim \mathscr{M}$ and $\mathscr{X}^* = (\mathscr{X}_0 \oplus \mathscr{K})^* = \mathscr{X}_0^* \oplus \mathscr{K}^*$, so $\operatorname{Ran} A = \mathscr{X}_0^*$, as claimed in Item (2). □

We have the following generalization of [33, Lemma D.3]; note that Lemma 2.3 (2) does not directly generalize Lemma 2.2 (2), since $\mathscr{X}$ is assumed to be reflexive in Lemma 2.3 and while it also helps motivate Definition 1.5 (1), it is not used elsewhere in this article.

Lemma 2.3 (Isomorphism properties of a symmetric operator)**.** *Let $\mathscr{X}$ be a reflexive Banach space over $\mathbb{K}$. If $A \in \mathscr{L}(\mathscr{X}, \mathscr{X}^*)$ is symmetric with closed range, then the following hold:*

(1) If $\operatorname{Ker} A = \{0\}$*, then* $\operatorname{Ran} A = \mathscr{X}^*$*;*
(2) If $\operatorname{Ker} A$ *has a closed complement $\mathscr{X}_0 \subset \mathscr{X}$, then* $\operatorname{Ran} A \cong \mathscr{X}_0^*$*.*

Proof. If $M \subset \mathscr{X}^*$ is a subspace, we recall from [73, Section 4.6] that its annihilator is

$$M^\perp := \{\phi \in \mathscr{X}^{**} : \langle \alpha, \phi \rangle = 0,\ \forall\, \alpha \in M\},$$

where $\langle \cdot, \cdot \rangle : \mathscr{X}^* \times \mathscr{X}^{**} \to \mathbb{K}$ denotes the canonical pairing, and that by [73, Theorem 4.12],

$$(\operatorname{Ran} A)^\perp = \operatorname{Ker} A^*, \tag{2.1}$$

where $A^* : \mathscr{X}^{**} \to \mathscr{X}^*$ is the adjoint operator defined by

$$\langle x, A^*\phi \rangle := \langle Ax, \phi \rangle, \quad \forall\, x \in \mathscr{X},\ \phi \in \mathscr{X}^{**}.$$

If $J : \mathscr{X} \to \mathscr{X}^{**}$ is the canonical map defined by $J(x)\alpha = \alpha(x)$ for all $x \in \mathscr{X}$ and $\alpha \in \mathscr{X}^*$, then J is an isomorphism by hypothesis that $\mathscr{X}$ is reflexive and thus

$$\langle y, A^*J(x) \rangle = \langle Ay, J(x) \rangle = \langle x, Ay \rangle, \quad \forall\, x, y \in \mathscr{X},$$

where $\langle \cdot, \cdot \rangle : \mathscr{X} \times \mathscr{X}^* \to \mathbb{K}$ also denotes the canonical pairing, that is,

$$\langle y, A^*J(x) \rangle = \langle x, Ay \rangle, \quad \forall\, x, y \in \mathscr{X}. \tag{2.2}$$

Hence,

$$\begin{aligned}
\operatorname{Ker} A^* &= \{\phi \in \mathscr{X}^{**} : \langle y, A^*\phi \rangle = 0,\ \forall\, y \in \mathscr{X}\} \\
&= J\left(\{x \in \mathscr{X} : \langle y, A^*J(x) \rangle = 0,\ \forall\, y \in \mathscr{X}\}\right) \quad \text{(by reflexivity of } \mathscr{X}\text{)} \\
&= J\left(\{x \in \mathscr{X} : \langle x, Ay \rangle = 0,\ \forall\, y \in \mathscr{X}\}\right) \quad \text{(by (2.2))} \\
&= J\left(\{x \in \mathscr{X} : \langle y, Ax \rangle = 0,\ \forall\, y \in \mathscr{X}\}\right) \quad \text{(by symmetry of } A\text{)},
\end{aligned}$$

that is,

$$\operatorname{Ker} A^* = J\left(\operatorname{Ker} A\right) \cong \operatorname{Ker} A. \tag{2.3}$$

Consider Item (1). Because $\operatorname{Ker} A = \{0\}$ by assumption, then (2.1) and (2.3) imply that

$$(\operatorname{Ran} A)^\perp = \{0\}. \tag{2.4}$$

If $\overline{\operatorname{Ran} A}$ denotes the (norm) closure of $\operatorname{Ran} A$ in $\mathscr{X}^*$, then

$$\begin{aligned}
\operatorname{Ran} A &= \overline{\operatorname{Ran} A} \quad \text{(as } \operatorname{Ran} A \text{ closed by hypothesis)} \\
&= {}^\perp\left((\operatorname{Ran} A)^\perp\right) \quad \text{(by [73, Theorem 4.7(a)])} \\
&= {}^\perp\{0\} \quad \text{(by (2.4))} \\
&= \mathscr{X}^*,
\end{aligned}$$

where if $N \subset X^{**}$ is a subspace, we recall from [73, Section 4.6] that its annihilator is

$${}^\perp N := \{\alpha \in \mathscr{X}^* : \langle \alpha, \phi \rangle = 0,\ \forall\, \phi \in N\}.$$

This establishes Item (1).

Consider Item (2). The argument yielding Item (1) now yields

$$\operatorname{Ran} A = {}^\perp\left(J\left(\operatorname{Ker} A\right)\right).$$

But

$$\begin{aligned}
{}^\perp\left(J\left(\operatorname{Ker} A\right)\right) &= \{\alpha \in \mathscr{X}^* : \langle \alpha, Jx \rangle = 0,\ \forall\, x \in \operatorname{Ker} A\} \\
&= \{\alpha \in \mathscr{X}^* : \langle x, \alpha \rangle = 0,\ \forall\, x \in \operatorname{Ker} A\} \\
&= \left(\operatorname{Ker} A\right)^\perp,
\end{aligned}$$

where if $L \subset \mathscr{X}$ is a subspace, we recall from [73, Section 4.6] that its annihilator is

$$L^\perp := \{\alpha \in \mathscr{X}^* : \langle x, \alpha \rangle = 0, \ \forall\, x \in L\},$$

where $\langle \cdot, \cdot \rangle : \mathscr{X} \times \mathscr{X}^* \to \mathbb{K}$ denotes the canonical pairing. Therefore, by combining the preceding identities, we obtain

$$\operatorname{Ran} A = (\operatorname{Ker} A)^\perp .$$

Since $\operatorname{Ker} A \subset \mathscr{X}$ is a closed subspace, the quotient space, $\mathscr{X} / \operatorname{Ker} A$, is a Banach space (by [15, Proposition 11.8]). By assumption, $\operatorname{Ker} A$ has a closed complement, $\mathscr{X}_0 \subset \mathscr{X}$, so $\mathscr{X} = \mathscr{X}_0 \oplus \operatorname{Ker} A$ and $\mathscr{X} / \operatorname{Ker} A \cong \mathscr{X}_0$. Hence, by [15, Proposition 11.9] there is an isomorphism of Banach spaces,

$$\mathscr{X}_0^* \cong (\operatorname{Ker} A)^\perp .$$

Consequently, we find that

$$\operatorname{Ran} A \cong \mathscr{X}_0^*,$$

as claimed. This establishes Item (2) and completes the proof of Lemma 2.3. □

We the following generalization of Lemma 2.2 which helps motivate Definition 1.8 (1) but is not used elsewhere in this article.

Lemma 2.4 (Isomorphism properties of a Fredholm operator)**.** *Let $\mathscr{X}$ and $\tilde{\mathscr{X}}$ be Banach spaces over $\mathbb{K}$. If $T \in \mathscr{L}(\mathscr{X}, \tilde{\mathscr{X}})$ is Fredholm and $\mathscr{X}_0$ is a closed complement of $\operatorname{Ker} T$ in $\mathscr{X}$, then the following hold:*

(1) $\operatorname{Ran} T \cong \mathscr{X}_0$ *and* $\tilde{\mathscr{X}} \cong \mathscr{X}_0 \oplus \operatorname{Ker} T^*$*;*
(2) If $\operatorname{Index} T = 0$*, then* $\tilde{\mathscr{X}} \cong \mathscr{X}_0 \oplus \operatorname{Ker} T$.

Proof. Consider Item (1). Since T is Fredholm, then $\operatorname{Ker} T$ is finite-dimensional and thus has a closed complement, $\mathscr{X}_0 \subset \mathscr{X}$, such that $\mathscr{X} = \mathscr{X}_0 \oplus \operatorname{Ker} T$ by [73, Lemma 4.21 (a)]. Similarly, because T is Fredholm, $\operatorname{Ran} T$ is a closed subspace of $\tilde{\mathscr{X}}$ and $\operatorname{Coker} T = \tilde{\mathscr{X}} / \operatorname{Ran} T$ is finite-dimensional, so $\operatorname{Ran} T$ has a closed complement, $\tilde{\mathscr{K}} \subset \tilde{\mathscr{X}}$ such that $\tilde{\mathscr{X}} = \operatorname{Ran} T \oplus \tilde{\mathscr{K}}$ by [73, Lemma 4.21 (b)], and $\tilde{\mathscr{X}} / \operatorname{Ran} T = \tilde{\mathscr{K}}$. Since $\operatorname{Ran} T$ is a Banach space and $T : \mathscr{X}_0 \to \operatorname{Ran} T$ bijective and bounded, then T is an isomorphism from $\mathscr{X}_0$ onto $\operatorname{Ran} T$ by the Open Mapping Theorem. By [15, Proposition 11.9], we have $\tilde{\mathscr{K}}^* = (\operatorname{Ran} T)^\perp$, and $(\operatorname{Ran} T)^\perp = \operatorname{Ker} T^*$ by [73, Theorem 4.12], and $\tilde{\mathscr{K}} \cong \tilde{\mathscr{K}}^*$ by finite-dimensionality. Hence, $\tilde{\mathscr{X}} \cong \mathscr{X}_0 \oplus \operatorname{Ker} T^*$, as claimed.

Consider Item (2). If $\operatorname{Index} T = 0$, then $\dim \operatorname{Ker} T^* = \dim \operatorname{Ker} T$ and $\operatorname{Ker} T \cong \operatorname{Ker} T^*$ by finite-dimensionality and $\tilde{\mathscr{X}} \cong \mathscr{X}_0 \oplus \operatorname{Ker} T$. □

2.2. **Preliminaries on nonlinear functional analysis.** In this subsection, we gather a few observations from nonlinear functional analysis.

2.2.1. *Differentiable and analytic maps on Banach spaces.* We refer to Huang [47, Section 2.1A]; see also Berger [9, Section 2.3]. Let $\mathscr{X}, \mathscr{Y}$ be Banach spaces over $\mathbb{K}$, let $\mathscr{U} \subset \mathscr{X}$ be an open subset, and $\mathscr{F} : \mathscr{U} \to \mathscr{Y}$ be a map. Recall that $\mathscr{F}$ is *Fréchet differentiable* at a point $x \in \mathscr{U}$ with a derivative, $\mathscr{F}'(x) \in \mathscr{L}(\mathscr{X}, \mathscr{Y})$, if

$$\lim_{y \to 0} \frac{1}{\|y\|_{\mathscr{X}}} \|\mathscr{F}(x+y) - \mathscr{F}(x) - \mathscr{F}'(x)y\|_{\mathscr{Y}} = 0.$$

Recall from Berger [9, Definition 2.3.1], Deimling [25, Definition 15.1], or Zeidler [92, Definition 8.8] that $\mathscr{F}$ is *analytic* at $x \in \mathscr{U}$ if there exists a constant $r > 0$ and a sequence of continuous symmetric linear maps, $L_n : \otimes^n \mathscr{X} \to \mathscr{Y}$, such that $\sum_{n\geq 1} \|L_n\| r^n < \infty$ and there is a positive constant $\delta = \delta(x)$ such that

$$\mathscr{F}(x+y) = \mathscr{F}(x) + \sum_{n\geq 1} L_n(y^n), \quad \|y\|_{\mathscr{X}} < \delta, \tag{2.5}$$

where $y^n \equiv (y, \dots, y) \in \mathscr{X} \times \cdots \times \mathscr{X}$ (n-fold product). If $\mathscr{F}$ is differentiable (respectively, analytic) at every point $x \in \mathscr{U}$, then $\mathscr{F}$ is differentiable (respectively, analytic) on $\mathscr{U}$. It is a useful observation that if $\mathscr{F}$ is analytic at $x \in \mathscr{X}$, then it is analytic on a ball $B_x(\varepsilon)$ (see Whittlesey [90, p. 1078]).

2.2.2. *Smooth and analytic inverse and implicit function theorems for maps on Banach spaces.* Statements and proofs of the Inverse Function Theorem for C^k maps of Banach spaces are provided by Abraham, Marsden, and Ratiu [1, Theorem 2.5.2], Deimling [25, Theorem 4.15.2], Zeidler [92, Theorem 4.F]; statements and proofs of the Inverse Function Theorem for *analytic* maps of Banach spaces are provided by Berger [9, Corollary 3.3.2] (complex), Deimling [25, Theorem 4.15.3] (real or complex), and Zeidler [92, Corollary 4.37] (real or complex). The corresponding C^k or Analytic Implicit Function Theorems are proved in the standard way as corollaries, for example [1, Theorem 2.5.7] and [92, Theorem 4.H].

2.2.3. *Gradient maps.* We recall the following basic facts concerning gradient maps.

Proposition 2.5 (Properties of gradient maps). *(See Huang* [47, Proposition 2.1.2]*.) Let $\mathscr{U}$ be an open subset of a Banach space, $\mathscr{X}$, let $\mathscr{Y}$ be continuously embedded in $\mathscr{X}^*$, and let $\mathscr{M} : \mathscr{U} \to \mathscr{Y} \subset \mathscr{X}^*$ be a continuous map. Then the following hold.*

(1) If $\mathscr{M}$ is a gradient map for $\mathscr{E}$, then

$$\mathscr{E}(x_1) - \mathscr{E}(x_0) = \int_0^1 \langle x_1 - x_0, \mathscr{M}(tx_1 + (1-t)x_0))\rangle_{\mathscr{X}\times\mathscr{X}^*}\, dt, \quad \forall\, x_0, x_1 \in \mathscr{U}.$$

(2) If $\mathscr{M}$ is of class C^1, then $\mathscr{M}$ is a gradient map if and only if all of its Fréchet derivatives, $\mathscr{M}'(x)$ for $x \in \mathscr{U}$, are symmetric in the sense that

$$\langle w, \mathscr{M}'(x)v\rangle_{\mathscr{X}\times\mathscr{X}^*} = \langle v, \mathscr{M}'(x)w\rangle_{\mathscr{X}\times\mathscr{X}^*}, \quad \forall\, x \in \mathscr{U} \text{ and } v, w \in \mathscr{X}.$$

(3) If $\mathscr{M}$ is an analytic gradient map, then any potential $\mathscr{E} : \mathscr{U} \to \mathbb{R}$ for $\mathscr{M}$ is analytic.

2.3. **Morse Lemma for functions on Banach spaces with degenerate critical points.** In this subsection, we prove Theorem 5 and hence Theorem 4 by taking $\tilde{\mathscr{X}} = \mathscr{X}^*$.

Given Banach spaces, $\mathscr{X}$ and $\mathscr{Z}$, over $\mathbb{K}$, an open subset $\mathscr{U} \subset \mathscr{X}$, and a smooth map, $f : \mathscr{U} \to \mathscr{Z}$, and an integer $n \geq 0$, we partly follow Zeidler [92, Sections 4.3–4.5] and let $D^n f(x) = f^{(n)}(x) \in \mathscr{L}^n(\mathscr{X}, \mathscr{Z}) = \mathscr{L}(\otimes^n \mathscr{X}, \mathscr{Z})$ denote the derivatives of order n at a point $x \in \mathscr{U}$. If $\mathscr{X} = \mathscr{X}_1 \times \mathscr{X}_2$, a product of Banach spaces $\mathscr{X}_i$ over $\mathbb{K}$ for $i = 1, 2$, we let $D_{x_i} f(x_1, x_2) = f_{x_i}(x_1, x_2) \in \mathscr{L}(\mathscr{X}_i, \mathscr{Z})$ and $D^2_{x_i x_j} f(x_1, x_2) = f_{x_i x_j}(x_1, x_2) \in \mathscr{L}(\mathscr{X}_i \otimes \mathscr{X}_j, \mathscr{Z})$ denote the first and second-order partial derivatives at a point $(x_1, x_2) \in \mathscr{U}$. We let $\mathrm{Iso}(\mathscr{X}) \subset \mathscr{L}(\mathscr{X})$ denote the open subgroup of invertible operators on $\mathscr{X}$.

Proof of Theorem 5. We generalize Ang and Tuan's proof of the [4, Morse–Palais Lemma], where f is C^{p+2} with $p \geq 1$ and $\mathscr{Y} = \{0\}$ and $0 \in \mathscr{X}$ is a non-degenerate critical point, Hörmander's proof of [45, Lemma C.6.1], where f is C^∞ and $(0,0) \in \mathscr{X} \times \mathscr{Y}$ is a degenerate critical point and $\mathscr{X} = \mathbb{R}^n$ and $\mathscr{Y} = \mathbb{R}^m$, and Lang's proof of [53, Theorem 7.5.1], where f is C^{p+2} with $p \geq 1$ and $\mathscr{Y} = \{0\}$ and $0 \in \mathscr{X}$ is a non-degenerate critical point and $\mathscr{X}$ is a real Hilbert space.

Consider the C^{p+1} map,

$$\mathscr{M} : \mathscr{X} \times \mathscr{Y} \supset \mathscr{U} \times \mathscr{V} \ni (x, y) \mapsto \mathscr{M}(x, y) := D_x f(x, y) \in \tilde{\mathscr{X}},$$

and observe that its partial derivative with respect to x, that is, the C^p map,

$$D_x\mathscr{M} : \mathscr{U} \times \mathscr{V} \times \mathscr{X} \ni (x, y, \eta) \mapsto D_x\mathscr{M}(x, y)\eta = D_x^2 f(x, y)\eta \in \tilde{\mathscr{X}},$$

gives an isomorphism, $\mathscr{X} \ni \eta \mapsto D_x\mathscr{M}(0, 0)\eta \in \tilde{\mathscr{X}}$ by our hypothesis on $D_x^2 f(0, 0) = D_x\mathscr{M}(0, 0)$. By the Implicit Function Theorem, after possibly shrinking $\mathscr{U}$ and $\mathscr{V}$, there exists a C^{p+1} map,

$$\psi : \mathscr{Y} \supset \mathscr{V} \ni y \mapsto w = \psi(y) \in \mathscr{U} \subset \mathscr{X},$$

with $\psi(0) = 0$, such that $\mathscr{M}(x, y) = 0$ if and only if $x = \psi(y)$, for each $y \in \mathscr{V}$; moreover, $D_y(\mathscr{M}(\psi(y), y)) = 0 = D_x\mathscr{M}(\psi(y), y)D_y\psi(y) + D_y\mathscr{M}(\psi(y), y)$ and so

$$D_y\psi(y) = -D_x\mathscr{M}(\psi(y), y)^{-1}D_y\mathscr{M}(\psi(y), y) \in \mathscr{L}(\mathscr{Y}, \mathscr{X}), \quad \forall\, y \in \mathscr{V}.$$

Define a C^{p+1} map, $\Psi : \mathscr{U} \times \mathscr{V} \ni (w, y) \mapsto (x, y) = \Psi(w, y) := (w + \psi(y), y) \in \mathscr{U} \times \mathscr{V}$, a C^{p+1} function $\tilde{f}$, and a C^{p+1} map, $\tilde{\mathscr{M}}$, by

$$\begin{aligned}
\tilde{f}(w, y) &:= f \circ \Psi(w, y) = f(w + \psi(y), y),\\
\tilde{\mathscr{M}}(w, y) &:= D_w\tilde{f}(w, y) = D_x f(w + \psi(y), y)\\
&\quad\; = \mathscr{M}(w + \psi(y), y), \quad \forall\, (w, y) \in \mathscr{U} \times \mathscr{V}.
\end{aligned}$$

The map, Ψ, has derivative,

$$D\Psi(w, y) = \begin{pmatrix} \mathrm{id}_{\mathscr{X}} & D\psi(y) \\ 0 & \mathrm{id}_{\mathscr{Y}} \end{pmatrix} \in \mathscr{L}(\mathscr{X} \oplus \mathscr{Y}), \quad \forall\, (w, y) \in \mathscr{U} \times \mathscr{V}, \tag{2.6}$$

an invertible operator at each $(w, y) \in \mathscr{U} \times \mathscr{V}$. In particular, after possibly shrinking $\mathscr{U}$ and $\mathscr{V}$, the map Ψ is a C^{p+1} diffeomorphism of an open neighborhood of the origin in $\mathscr{X} \times \mathscr{Y}$ by the Inverse Function Theorem. Therefore, the identity

$$\mathscr{M}(\psi(y), y) = 0, \quad \forall\, y \in \mathscr{V},$$

is equivalent to

$$\tilde{\mathscr{M}}(0, y) = 0, \quad \forall\, y \in \mathscr{V},$$

since $\tilde{\mathscr{M}}(0, y) = \mathscr{M}(\psi(y), y) \circ D\Psi(0, y)$ and $D\Psi(0, y)$ is invertible, for all $y \in \mathscr{V}$. The Chain Rule gives

$$D_w^2\tilde{f}(w, y) = D_x^2 f(x, y) \quad \text{and thus} \quad D_w^2\tilde{f}(0, 0) = D_x^2 f(0, 0) = A.$$

By shrinking $\mathscr{U}$ if necessary, we may assume that $\mathscr{U}$ is convex and so by the second-order Taylor Formula we have

$$\tilde{f}(w, y) = \tilde{f}(0, y) + D_w\tilde{f}(0, y)w + \int_0^1 (1 - t)D_w^2\tilde{f}(tw, y)w^2\, dt, \quad \forall\, (w, y) \in \mathscr{U} \times \mathscr{V},$$

that is,

$$\tilde{f}(w, y) = \tilde{f}(0, y) + \langle w, \tilde{\mathscr{M}}(0, y)\rangle + \int_0^1 (1 - t)\langle w, D_w\tilde{\mathscr{M}}(tw, y)w\rangle\, dt, \quad \forall\, (w, y) \in \mathscr{U} \times \mathscr{V}. \tag{2.7}$$

Therefore, using $\tilde{\mathscr{M}}(0, y) = 0$ for all $y \in \mathscr{V}$,

$$\tilde{f}(w, y) = \tilde{f}(0, y) + \frac{1}{2}\langle w, B(w, y)w\rangle, \quad \forall\, (w, y) \in \mathscr{U} \times \mathscr{V},$$

where

$$B(w, y) := 2\int_0^1 (1 - t)D_w\tilde{\mathscr{M}}(tw, y)\, dt, \quad \forall\, (w, y) \in \mathscr{U} \times \mathscr{V}.$$

The expression for B defines a C^p map,

$$\mathscr{X} \times \mathscr{Y} \supset \mathscr{U} \times \mathscr{V} \ni (w,y) \mapsto B(w,y) \in \mathscr{L}_{\text{sym}}(\mathscr{X}, \tilde{\mathscr{X}}),$$

such that $B(0,0) = D_w\tilde{\mathscr{M}}(0,0) = A$.

We wish to write $z = R(w,y)w \in \mathscr{X}$ where, after possibly further shrinking $\mathscr{U}$ and $\mathscr{V}$,

$$\mathscr{X} \times \mathscr{Y} \supset \mathscr{U} \times \mathscr{V} \ni (w,y) \mapsto R(w,y) \in \operatorname{Iso}(\mathscr{X}) \cap \mathscr{L}_A(\mathscr{X})$$

is a C^p map such that $R(0,0) = \mathrm{id}_{\mathscr{X}}$ and

$$\langle w, B(w,y)w\rangle = \langle R(w,y)w, AR(w,y)w\rangle, \quad \forall\,(w,y) \in \mathscr{U} \times \mathscr{V}. \tag{2.8}$$

The identity (2.8) follows if we can write

$$B(w,y) = R(w,y)^*AR(w,y), \quad \forall\,(w,y) \in \mathscr{U} \times \mathscr{V}. \tag{2.9}$$

Equation (2.9) is valid at $(w,y) = (0,0)$ with $R(0,0) = \mathrm{id}_{\mathscr{X}}$ and $B(0,0) = A$.

We now generalize an argument due to Ang and Tuan (see [4, Lemma 1]) from the case $\tilde{\mathscr{X}} = \mathscr{X}^*$ to the case $\tilde{\mathscr{X}} \subset \mathscr{X}^*$ and make the

Definition 2.6 (A closed subspace of the space of continuous, linear operators)**.** Let $\mathscr{L}_A(\mathscr{X}) \subset \mathscr{L}(\mathscr{X})$ denote the closed subspace of operators $R \in \mathscr{L}(\mathscr{X})$ whose adjoints $R^* \in \mathscr{L}(\mathscr{X}^*)$ restrict to[8] operators $R^* \restriction \tilde{\mathscr{X}} \in \mathscr{L}(\tilde{\mathscr{X}})$ after composition with the embedding $\tilde{\mathscr{X}} \subset \mathscr{X}^*$ and obey

$$R^*A = AR \in \mathscr{L}(\mathscr{X}, \tilde{\mathscr{X}}). \tag{2.10}$$

We have the following generalization of Ang and Tuan [4, Lemma 1].

Claim 2.7 (Isomorphism onto a space of continuous, linear symmetric operators)**.** *The following linear map is an isomorphism of Banach spaces,*

$$\mathscr{L}_A(\mathscr{X}) \ni Q \mapsto Q^*A + AQ \in \mathscr{L}_{\text{sym}}(\mathscr{X}, \tilde{\mathscr{X}}). \tag{2.11}$$

Proof. We first observe that the map (2.11) is well-defined by virtue of the Definition 2.6 of the subspace $\mathscr{L}_A(\mathscr{X})$. Second, we show that the map (2.11) is surjective. If $C \in \mathscr{L}_{\text{sym}}(\mathscr{X}, \tilde{\mathscr{X}})$, set

$$Q := \frac{1}{2}A^{-1}C \in \mathscr{L}(\mathscr{X}).$$

The adjoint of Q is $Q^* = \frac{1}{2}C^*(A^{-1})^* \in \mathscr{L}(\mathscr{X}^*)$. Now, $A^* = A$ and $C^* = C \in \mathscr{L}(\mathscr{X}, \tilde{\mathscr{X}})$ by our earlier discussion of properties of operators in $\mathscr{L}_{\text{sym}}(\mathscr{X}, \tilde{\mathscr{X}})$ and thus also $(A^*)^{-1} = A^{-1} \in \mathscr{L}(\tilde{\mathscr{X}}, \mathscr{X})$. But[9] $(A^{-1})^* = (A^*)^{-1} \in \mathscr{L}(\mathscr{X}^*, \tilde{\mathscr{X}}^*)$ and thus $(A^{-1})^* = A^{-1} \in \mathscr{L}(\tilde{\mathscr{X}}, \mathscr{X})$. By combining these observations, we see that $Q^* = \frac{1}{2}C^*(A^{-1})^* = \frac{1}{2}CA^{-1} \in \mathscr{L}(\tilde{\mathscr{X}})$, so $Q \in \mathscr{L}_A(\mathscr{X})$, as required, and

$$Q^*A + AQ = \frac{1}{2}\left(C^*(A^{-1})^*A + AA^{-1}C\right) = \frac{1}{2}\left(CA^{-1}A + AA^{-1}C\right) = C \in \mathscr{L}_{\text{sym}}(\mathscr{X}, \tilde{\mathscr{X}}).$$

completing the proof of surjectivity. Third, we show that the map (2.11) is injective. If $AQ + Q^*A = 0$, then $AQ = -Q^*A \in \mathscr{L}(\mathscr{X}, \tilde{\mathscr{X}})$ while $AQ = Q^*A$ by (2.10) and thus $AQ = 0$ and so $Q = 0 \in \mathscr{L}_A(\mathscr{X})$ since A is invertible. Clearly, the map (2.11) is continuous and its inverse is also continuous by the Open Mapping Theorem. This completes the proof of Claim 2.7. □

[8] To avoid notational clutter, we omit explicit notation, such as $\iota : \tilde{\mathscr{X}} \subset \mathscr{X}^*$, for the continuous embedding.

[9] Because $AA^{-1} = \mathrm{id}_{\mathscr{X}} = A^{-1}A$ and by [73, Exercise 4.8], one has $(A^{-1})^*A^* = \mathrm{id}_{\mathscr{X}} = A^*(A^{-1})^*$, so $(A^*)^{-1} = (A^{-1})^*$.

The derivative of the quadratic map,

$$\mathscr{Q} : \mathscr{L}_A(\mathscr{X}) \ni P \mapsto P^*AP \in \mathscr{L}_{\mathrm{sym}}(\mathscr{X}, \tilde{\mathscr{X}}), \tag{2.12}$$

at P in the direction Q is given by

$$D\mathscr{Q}(P) : \mathscr{L}_A(\mathscr{X}) \ni Q \mapsto Q^*AP + P^*AQ \in \mathscr{L}_{\mathrm{sym}}(\mathscr{X}, \tilde{\mathscr{X}}). \tag{2.13}$$

Note that the map (2.12) is well-defined. Indeed, $(P^*AP)^* = P^*A^*P^{**} \in \mathscr{L}(\mathscr{X}^{**})$, where $P \in \mathscr{L}(\mathscr{X})$ has adjoint operator $P^* \in \mathscr{L}(\mathscr{X}^*)$ and bidual operator $P^{**} \in \mathscr{L}(\mathscr{X}^{**})$. But $P^{**} \restriction \mathscr{X} = P$ (for example, see Brezis [15, Theorem 3.24] or Pietsch [69, Chapter 0, Section A.3.6]) and thus $(P^*AP)^* = P^*A^*P = P^*AP \in \mathscr{L}(\mathscr{X}, \tilde{\mathscr{X}})$ and P^*AP is symmetric. When P is the identity operator, we have $D\mathscr{Q}(\mathrm{id}_{\mathscr{X}}) = Q^*A + AQ$ and this operator is an isomorphism by Claim 2.7.

We now adapt the proof of Ang and Tuan [4, Lemma 2] and the remainder of the proof of Hörmander [45, Lemma C.6.1]. The Analytic Implicit Function Theorem[10] provides open neighborhoods, $\mathscr{O}_{\mathrm{id}} \subset \mathscr{L}_A(\mathscr{X})$ of the identity operator $\mathrm{id}_{\mathscr{X}}$ and $\mathscr{O}_A \subset \mathscr{L}_{\mathrm{sym}}(\mathscr{X}, \tilde{\mathscr{X}})$ of the operator A, such that the restriction of the analytic map (2.12),

$$\mathscr{L}_A(\mathscr{X}) \supset \mathscr{O}_{\mathrm{id}} \ni Q \mapsto Q^*AQ \in \mathscr{O}_A \subset \mathscr{L}_{\mathrm{sym}}(\mathscr{X}, \tilde{\mathscr{X}}),$$

is an analytic diffeomorphism onto its image, with analytic inverse,

$$F : \mathscr{L}_{\mathrm{sym}}(\mathscr{X}, \tilde{\mathscr{X}}) \supset \mathscr{O}_A \ni S \mapsto F(S) \in \mathscr{O}_{\mathrm{id}} \subset \mathscr{L}_A(\mathscr{X}),$$

such that $F(A) = \mathrm{id}_{\mathscr{X}}$. Therefore, Equation (2.8) is fulfilled when we choose $R = F(B)$. Substituting $z = R(w,y)w$ in Equation (2.8) and combining this identity with our previous expression (2.7) for $\tilde{f}(w,y)$ yields

$$\begin{aligned} \tilde{f}(w,y) &= \tilde{f}(0,y) + \frac{1}{2}\langle w, B(w,y)w\rangle \\ &= \tilde{f}(0,y) + \frac{1}{2}\langle R(w,y)w, AR(w,y)w\rangle \\ &= \tilde{f}(0,y) + \frac{1}{2}\langle z, Az\rangle, \quad \forall\, (w,y) \in \mathscr{U} \times \mathscr{V}. \end{aligned}$$

Observe that the C^p map, $\mathscr{U} \times \mathscr{V} \ni (w,y) \mapsto (R(w,y)w, y) \in \mathscr{X} \times \mathscr{Y}$, has derivative at the origin,

$$\begin{pmatrix} \mathrm{id}_{\mathscr{X}} & 0 \\ 0 & \mathrm{id}_{\mathscr{Y}} \end{pmatrix} \in \mathscr{L}(\mathscr{X} \oplus \mathscr{Y}), \tag{2.14}$$

since $R(0,0) = \mathrm{id}_{\mathscr{X}}$, and thus is invertible. Hence, after possibly further shrinking $\mathscr{U}$ and applying the Inverse Function Theorem, the map

$$\mathscr{U} \times \mathscr{V} \ni (w,y) \mapsto (z,y) = (R(w,y)w, y) \in \mathscr{U}' \times \mathscr{V},$$

is a C^p diffeomorphism onto $\mathscr{U}' \times \mathscr{V}$, where $\mathscr{U}'$ is an open neighborhood of the origin in $\mathscr{X}$. We denote its C^p inverse map by

$$\Xi : \mathscr{U}' \times \mathscr{V} \ni (z,y) \mapsto (w,y) \in \mathscr{U} \times \mathscr{V},$$

and note that $\Xi(0,0) = (0,0)$ with derivative at the origin,

$$D\Xi(0,0) = \begin{pmatrix} \mathrm{id}_{\mathscr{X}} & 0 \\ 0 & \mathrm{id}_{\mathscr{Y}} \end{pmatrix} \in \mathscr{L}(\mathscr{X} \oplus \mathscr{Y}), \tag{2.15}$$

[10] Lang [53, Theorem 5.2] and Palais [66, p. 969] use a power series argument to define F rather than apply the Analytic Implicit Function Theorem.

by (2.14). Consequently,

$$\tilde{f}(\Xi(z,y)) = \tilde{f}(0,y) + \frac{1}{2}\langle z, Az\rangle, \quad \forall\, (z,y) \in \mathscr{U}' \times \mathscr{V}.$$

But $\tilde{f}(w,y) = f(\Psi(w,y))$ and setting $(x,y) = \Psi(w,y) = \Psi(\Xi(z,y)) =: \Phi(z,y)$, we obtain

$$f(\Phi(z,y)) = f(\Phi(0,y)) + \frac{1}{2}\langle z, Az\rangle, \quad \forall\, (z,y) \in \mathscr{U}' \times \mathscr{V},$$

which is the desired relation (1.12). Equations (2.6) and (2.15) and the Chain Rule give

$$D\Phi(0,0) = \begin{pmatrix} \mathrm{id}_{\mathscr{X}} & \star \\ 0 & \mathrm{id}_{\mathscr{Y}} \end{pmatrix} \in \mathscr{L}(\mathscr{X} \oplus \mathscr{Y}),$$

which is (1.11). The conclusion on analyticity of Φ follows by replacing the role of the Inverse Function Theorem for C^p maps in the preceding arguments by its counterpart for analytic maps when f is analytic (see Section 2.2.2). The proof of Theorem 5 is complete. □

2.4. **Applications to proofs of the Morse and Morse–Bott Lemmas for functions on Banach spaces.** We begin by recalling the

Theorem 2.8 (Morse Lemma for functions on Banach spaces with non-degenerate critical points)**.** *(See Palais* [65, p. 307], [66, p. 968]*.) Let $\mathscr{X}$ be a Banach space over $\mathbb{K}$, and $\mathscr{U} \subset \mathscr{X}$ be an open neighborhood of the origin, and $f : \mathscr{X} \supset \mathscr{U} \ni x \mapsto f(x) \in \mathbb{K}$ be a C^{p+2} function ($p \geq 1$) such that $f(0) = 0$ and $f'(0) = 0$. If $f''(0) \in \mathscr{L}(\mathscr{X}, \mathscr{X}^*)$ is invertible*[11] *then there are an open neighborhood of the origin, $\mathscr{V} \subset \mathscr{X}$, and a C^p diffeomorphism, $\mathscr{V} \ni y \mapsto x = \Phi(y) \in \mathscr{X}$ with $\Phi(0) = 0$ and $D\Phi(0) = \mathrm{id}_{\mathscr{X}}$, such that*

$$f(\Phi(z)) = \frac{1}{2}\langle z, Az\rangle, \quad \forall\, z \in \mathscr{U}, \tag{2.16}$$

where $A := f''(0) = (f \circ \Phi)''(0) \in \mathscr{L}_{\mathrm{sym}}(\mathscr{X}, \mathscr{X}^*)$. If f is analytic, then Φ is analytic. If $\mathscr{X}$ is a Hilbert space with norm $\|\cdot\|$, then one may further choose Φ and an orthogonal decomposition, $\mathscr{X} = \mathscr{X}_+ \oplus \mathscr{X}_-$, such that

$$f(\Phi(z)) = \frac{1}{2}\left(\|z_+\|^2 - \|z_-\|^2\right), \quad \forall\, z = z_+ + z_- \in \mathscr{U}. \tag{2.17}$$

Theorem 2.8 is an immediate consequence of the more general Theorem 2.10 and which is proved below (see also Lang [53, Corollary 5.3]).

Remark 2.9 (Tangent space to the critical set as a subspace of the kernel of the Hessian operator). If the critical set $\operatorname{Crit} f$ of a smooth function $f : \mathscr{U} \to \mathbb{K}$ is a smooth submanifold of $\mathscr{U}$ and $x_0 \in \operatorname{Crit} f$, then $T_0 \operatorname{Crit} f \subset \operatorname{Ker} f''(x_0)$. Indeed, if $v \in T_0 \operatorname{Crit} f$ and $\gamma(t)$ is a smooth curve in $\operatorname{Crit} f$ with $\gamma(0) = x_0$ and $\gamma'(0) = v$, where $t \in (-\varepsilon, \varepsilon)$, then $f'(\gamma(t)) = 0 \in \mathscr{X}^*$, since $\gamma(t) \in \operatorname{Crit} f$, and so the Chain Rule gives

$$(f \circ \gamma)''(t) = f''(\gamma(t))\gamma'(t)^2 = 0.$$

Thus at $t = 0$, we have $f(\gamma(0))''\gamma'(0) = f''(x_0)v = 0 \in \mathscr{X}^*$ and hence $v \in \operatorname{Ker} f''(x_0)$.

We have the following generalization of Theorem 2.8.

[11] In other words, f is Morse at the point $0 \in \mathscr{X}$.

Theorem 2.10 (Morse–Bott Lemma for functions on Banach spaces)**.** *Let $\mathscr{X}$ be a Banach space over $\mathbb{K}$, and $\mathscr{U} \subset \mathscr{X}$ be an open neighborhood of the origin, and $f : \mathscr{U} \to \mathbb{K}$ be a C^{p+2} function ($p \geq 1$) such that $f(0) = 0$. If f is Morse–Bott at the origin in the sense of Definition 1.5 (1), then, after possibly shrinking $\mathscr{U}$, there are an open neighborhood of the origin, $\mathscr{V} \subset \mathscr{X}$, and a C^p diffeomorphism, $\mathscr{V} \ni y \mapsto x = \Phi(y) \in \mathscr{U}$ with $\Phi(0) = 0$ and $D\Phi(0) = \mathrm{id}_{\mathscr{X}}$, such that*

$$f(\Phi(y)) = \frac{1}{2}\langle y, Ay\rangle, \quad \forall\, y \in \mathscr{V}, \tag{2.18}$$

where $A := f''(0) = (f \circ \Phi)''(0) \in \mathscr{L}_{\mathrm{sym}}(\mathscr{X}, \mathscr{X}^*)$. If f is analytic, then Φ is analytic.

Remark 2.11 (Morse–Bott Lemma for functions on Banach spaces and local coordinates). By Definition 1.5 (1, the closed subspace, $\mathscr{K} = \operatorname{Ker} f''(0) \subset \mathscr{X}$, has a closed complement, $\mathscr{X}_0 \subset \mathscr{X}$, such that $\mathscr{X} = \mathscr{X}_0 \oplus \mathscr{K}$ (and so[12] $\mathscr{X}^* = \mathscr{X}_0^* \oplus \mathscr{K}^*$). If $\pi \in \mathscr{L}(\mathscr{X}, \mathscr{X}_0)$ and $\iota^* \in \mathscr{L}(\mathscr{X}^*, \mathscr{X}_0^*)$ are the continuous projections (where $\iota : \mathscr{X}_0 \to \mathscr{X}$ is the continuous injection), then (2.18) becomes

$$f(\Phi(y)) = \frac{1}{2}\langle \pi y, A_0 \pi y\rangle, \quad \forall\, y \in \mathscr{V},$$

where $A_0 := \iota^* A \pi \in \mathscr{L}_{\mathrm{sym}}(\mathscr{X}_0, \mathscr{X}_0^*)$ is an isomorphism. Indeed, if we write $x = (w, \xi) \in \mathscr{X}_0 \oplus \mathscr{K}$, then $y = \Phi(x) = (z, \xi) \in \mathscr{X}_0 \oplus \mathscr{K}$ for all $x \in \mathscr{U}$ and $A_0 = D_w^2 f(0,0) = D_z^2 (f \circ \Phi)(0,0)$ and (2.18) becomes

$$f(\Phi(z,\xi)) = \frac{1}{2}\langle z, A_0 z\rangle, \quad \forall\,(z,\xi) \in \mathscr{V} \cap (\mathscr{X}_0 \oplus \mathscr{K}),$$

for coordinates adapted to the direct sum decomposition.

Remark 2.12 (Morse–Bott Lemma for functions on Hilbert spaces). Suppose now that $\mathscr{X}$ is a Hilbert space and identify $\mathscr{X}^* \cong \mathscr{X}$, so $A \in \mathscr{L}(\mathscr{X})$ is self-adjoint (since $A \in \mathscr{L}(\mathscr{X}, \mathscr{X}^*)$ is symmetric) and thus has spectrum $\sigma(A) \subset \mathbb{R}$ by [73, Theorem 12.15 (b)]. By the Spectral Theorem for bounded normal operators on a Hilbert space [73, pp. 321–327], there are an orthogonal decomposition into closed invariant subspaces, $\mathscr{X} = \mathscr{X}_{0,+} \oplus \mathscr{X}_{0,-} \oplus \mathscr{K}$ corresponding to the Borel subsets, $(0,\infty)$, $(-\infty,0)$, and $\{0\}$ of $\sigma(A)$, continuous projections, $\pi_\pm \in \mathscr{L}(\mathscr{X}, \mathscr{X}_{0,\pm})$, and injections, $\iota_\pm \in \mathscr{L}(\mathscr{X}_{0,\pm}, \mathscr{X})$, and invertible positive operators, $A^+ := \pi_+ A \iota_+ \in \mathscr{L}(\mathscr{X}_{0,+})$ and $A^- := -\pi_- A \iota_- \in \mathscr{L}(\mathscr{X}_{0,-})$, so that

$$f(\Phi(z,\xi)) = \frac{1}{2}\langle z_+, A^+ z_+\rangle - \frac{1}{2}\langle z_-, A^- z_-\rangle, \quad \forall\,(z,\xi) \in \mathscr{V} \cap (\mathscr{X}_0 \oplus \mathscr{K}), \tag{2.19}$$

where $z_\pm = \pi_\pm z$. The operators $A^\pm$ have (unique) invertible positive square roots $S^\pm$ [73, Theorem 12.33] and so we may define a norm on $\mathscr{X}_0$ that is equivalent to $\|\cdot\|$ by setting $\|z^\pm\|_S = \|S^\pm z_\pm\|$ for all $z_\pm \in \mathscr{X}_{0,\pm}$, so that (2.19) becomes

$$f(\Phi(z,\xi)) = \frac{1}{2}\left(\|z^+\|_S^2 - \|z^-\|_S^2\right), \quad \forall\,(z,\xi) \in \mathscr{V} \cap (\mathscr{X}_0 \oplus \mathscr{K}), \tag{2.20}$$

as asserted in the special case ($\mathscr{K} = 0$) provided by Theorem 2.8.

Remark 2.13 (Expositions of the proofs of the Morse and Morse–Bott Lemmas for functions on Euclidean space). Nicolaescu provides a proof [63, Theorem 1.12] of the Morse Lemma for C^∞ functions on Euclidean space (Theorem 2.8 with $\mathscr{X} = \mathbb{R}^d$) based on that of Arnold, Gusein-Zade, and Varchenko [6, Section 6.4] and remarks that his proof extends to yield the Morse–Bott Lemma for C^∞ functions on Euclidean space (Theorem 2.10 with $\mathscr{X} = \mathbb{R}^d$) in [63, Proposition 2.42]). See Banyaga and Hurtubise [8, Theorem 2] for a recent exposition of the proof of the Morse–Bott Lemma for C^2 functions on Euclidean space.

[12]By Lemma 2.1.

We turn to the more general case where the derivative of f is represented by a gradient map.

Theorem 2.14 (Generalized Morse–Bott Lemma for functions on Banach spaces)**.** *Let $\mathscr{X}$ and $\tilde{\mathscr{X}}$ be Banach spaces over $\mathbb{K}$ with continuous embedding, $\tilde{\mathscr{X}} \subset \mathscr{X}^*$, and $\mathscr{U} \subset \mathscr{X}$ be an open neighborhood of the origin, and $f : \mathscr{U} \to \mathbb{K}$ be a C^{p+1} function ($p \geq 1$) such that $f(0) = 0$. If f is Morse–Bott at the origin in the sense of Definition 1.8 (1), then, after possibly shrinking $\mathscr{U}$, there are an open neighborhood of the origin, $\mathscr{V} \subset \mathscr{X}$, and a C^p diffeomorphism, $\mathscr{V} \ni y \mapsto x = \Phi(y) \in \mathscr{U}$ with $\Phi(0) = 0$, such that*

$$f(\Phi(y)) = \frac{1}{2}\langle y, Ay\rangle, \quad \forall\, y \in \mathscr{V}, \tag{2.21}$$

where $A := f''(0) = (f \circ \Phi)''(0) \in \mathscr{L}_{\mathrm{sym}}(\mathscr{X}, \tilde{\mathscr{X}})$ and, for $\mathscr{X} = \mathscr{X}_0 \oplus \mathscr{K}$ with $\mathscr{K} := \operatorname{Ker} f''(0)$ and closed complement $\mathscr{X}_0$,

$$D\Phi(0,0) = \begin{pmatrix} \mathrm{id}_{\mathscr{X}_0} & \star \\ 0 & \mathrm{id}_{\mathscr{K}} \end{pmatrix} \in \mathscr{L}(\mathscr{X}_0 \oplus \mathscr{K}). \tag{2.22}$$

If f is analytic, then Φ is analytic.

Proofs of Theorem 2.10 and 2.14. Observe that Theorem 2.10 follows immediately from Theorem 2.14 by restricting to the case $\tilde{\mathscr{X}} = \mathscr{X}^*$, so we focus on the more general case.

Because f is C^{p+2} and Morse–Bott at the origin, $\operatorname{Crit} f \subset \mathscr{U}$ is a C^2 submanifold by Definition 1.8 (1) and thus a C^{p+2} submanifold by the Implicit Function Theorem. Moreover, by Definition 1.8 (1), there is a direct sum decomposition, $\mathscr{X} = \mathscr{X}_0 \oplus \mathscr{K}$, where $\mathscr{K} = \operatorname{Ker} f''(0)$ and $\mathscr{X}_0$ is a closed complement and $T_0 \operatorname{Crit} f = \mathscr{K}$. Hence, after possibly shrinking $\mathscr{U}$, the Implicit Function Theorem provides a C^{p+2} diffeomorphism, Ξ, from an open neighborhood $\mathscr{O}$ of the origin in $\mathscr{X}$ onto $\mathscr{U}$ such that $\Xi(0) = 0$ and $D\Xi(0) = \mathrm{id}_{\mathscr{X}}$ with

$$\operatorname{Crit} f \circ \Xi = \mathscr{O} \cap (\{0\} \oplus \mathscr{K}).$$

Therefore, we may assume without loss of generality that

$$\operatorname{Crit} f = \mathscr{U} \cap (\{0\} \oplus \mathscr{K}).$$

Furthermore, Definition 1.8 (1) provides that $\operatorname{Ran} f''(0) = \tilde{\mathscr{X}}$. Hence, Theorem 5 implies that, after possibly shrinking $\mathscr{U}$, there exists a C^p diffeomorphism, $\Phi : \mathscr{U} \cap (\mathscr{X}_0 \oplus \mathscr{K}) \ni (z,\xi) \mapsto x = \Phi(z,\xi) \in \mathscr{X} = \mathscr{X}_0 \oplus \mathscr{K}$, such that $\Phi(0,0) = 0$ and $D\Phi(0,0)$ is as in (2.22) with

$$f(\Phi(z,\xi)) = \frac{1}{2}\langle z, A_0 z\rangle + g(\xi), \quad \forall\,(z,\xi) \in \mathscr{U} \cap (\mathscr{X}_0 \oplus \mathscr{K}),$$

where $g(\xi) := f(\Phi(0,\xi))$, and $A_0 := D^2 f(0) \restriction \mathscr{X}_0 = D_z^2(f \circ \Phi)(0,0)$ and $A_0 \in \mathscr{L}_{\mathrm{sym}}(\mathscr{X}_0, \tilde{\mathscr{X}})$ is an isomorphism by the Open Mapping Theorem. We observe that $D(f\circ\Phi)(z,\xi) = A_0 z + Dg(\xi) \in \tilde{\mathscr{X}}$. Hence, $(z,\xi) \in \operatorname{Crit} f \circ \Phi \iff z = 0$ and $Dg(\xi) = 0$, that is $\xi \in \operatorname{Crit} g$, where $g : \mathscr{U} \cap \mathscr{K} \to \mathbb{K}$ is a C^p function with $g(0) = 0$. Therefore, $\operatorname{Crit} f \circ \Phi = \operatorname{Crit} g$. In particular, $\operatorname{Crit} g$ is a C^p submanifold of $\mathscr{U} \cap \mathscr{K}$, since $\operatorname{Crit} f \circ \Phi$ is a C^{p+2} submanifold of $\mathscr{U}$, and $\dim \operatorname{Crit} f \circ \Phi = \dim \operatorname{Crit} g$ with $T_0 \operatorname{Crit} g = T_0 \operatorname{Crit} f \circ \Phi = \mathscr{K}$. Since $0 \in \operatorname{Crit} g$ and $g(0) = 0$, there is a connected open neighborhood of the origin in $\mathscr{K}$ such that $g \equiv 0$ and by shrinking $\mathscr{U}$ if necessary, we may assume that $g \equiv 0$ on $\mathscr{U} \cap \mathscr{K}$. Hence, $D(f \circ \Phi)(z,\xi) = A_0 z = A(z,\xi)$ by writing $A \in \mathscr{L}(\mathscr{X}_0 \oplus \mathscr{K}, \tilde{\mathscr{X}})$ as

$$A(z,\xi) = A_0 z, \quad \forall (z,\xi) \in \mathscr{X}_0 \oplus \mathscr{K}.$$

If f is analytic, then Φ is analytic. $\square$

When $\mathscr{X} = \mathbb{C}^d$, then Theorem 2.10 yields the

Corollary 2.15 (Holomorphic Morse–Bott Lemma for functions on $\mathbb{C}^d$)**.** *(See* [70] *for a statement in the case $c = 0$ and Petro* [68, Lemma 3.8] *for a statement in the case $c \geq 0$; compare Seidel* [74, Lemma 1.6]*.) Let $d \geq 2$ be an integer, $U \subset \mathbb{C}^d$ be an open neighborhood of the origin, and $f : U \ni x \mapsto f(x) \in \mathbb{C}$ be a holomorphic function such that $f(0) = 0$ and $f'(0) = 0$. Assume that $\operatorname{Crit} f$ is a complex submanifold of U with complex tangent space $T_0 \operatorname{Crit} f = \operatorname{Ker} f''(0)$ of dimension $c \geq 0$ at the origin. Then, after possibly shrinking U, there are an open neighborhood $V \subset \mathbb{C}^d$ of the origin and a complex analytic diffeomorphism, $V \ni (w_1, \dots, w_d) \mapsto (x_1, \dots, x_d) = \Phi(w_1, \dots, w_d) \in \mathbb{C}^d$, onto an open neighborhood of the origin in $\mathbb{C}^d$ such that*

$$\Phi^{-1}(U \cap \operatorname{Crit} \mathscr{E}) = V \cap (\mathbb{C}^c \cap 0) \subset \mathbb{C}^c \times \mathbb{C}^{d-c}$$

with $\Phi(0) = 0$ and

$$D\Phi(0) = \begin{pmatrix} \mathrm{id}_{d-c} & \star \\ 0 & \mathrm{id}_c \end{pmatrix} \in \mathrm{GL}(d, \mathbb{C}),$$

where $\mathrm{id}_{d-c} \in \mathrm{GL}(d-c, \mathbb{C})$ and $\mathrm{id}_c \in \mathrm{GL}(c, \mathbb{C})$ and

$$f(\Phi(w_1, \dots, w_d)) = w_1^2 + \cdots + w_{d-c}^2, \quad \forall\, w = (w_1, \dots, w_d) \in U. \tag{2.23}$$

3. Łojasiewicz gradient inequality for functions on Banach spaces

In Section 3.1, we use the Morse–Bott Lemma for C^{p+2} functions ($p \geq 1$) (see Theorems 2.10 and 2.14) to give a concise proof of the Łojasiewicz gradient inequality for C^{p+2} Morse–Bott functions on Banach spaces (see Theorems 6 and 7); in Section 3.2, we apply the Morse Lemma for analytic functions with degenerate critical points (see Theorems 4 and 5) to give an elegant proof of the Łojasiewicz gradient inequality for analytic functions on Banach spaces (see Theorems 8 and 9).

3.1. Łojasiewicz gradient inequality for smooth Morse–Bott functions.

In this subsection, we prove Theorem 7, and hence Theorem 6 upon choosing $\tilde{\mathscr{X}} = \mathscr{X}^*$. We begin with the

Lemma 3.1 (Łojasiewicz gradient inequality for quadratic forms)**.** *Let $\mathscr{X}$ and $\tilde{\mathscr{X}}$ be Banach spaces over $\mathbb{K}$ with continuous embedding, $\tilde{\mathscr{X}} \subset \mathscr{X}^*$. If $Q : \mathscr{X} \ni x \mapsto Q(x) = \frac{1}{2}\langle x, Ax\rangle \in \mathbb{K}$ is defined by a symmetric operator, $A \in \mathscr{L}_{\mathrm{sym}}(\mathscr{X}, \tilde{\mathscr{X}})$, whose kernel is complemented in $\mathscr{X}$ and whose range is $\tilde{\mathscr{X}}$, then Q has Łojasiewicz exponent $1/2$, that is, there is a constant $C \in (0, \infty)$ such that*

$$\|Q'(x)\|_{\tilde{\mathscr{X}}} \geq C Q(x)^{1/2}, \quad \forall\, x \in \mathscr{X}. \tag{3.1}$$

Proof. The derivative of $Q : \mathscr{X} \to \mathbb{K}$ is given by

$$Q'(x)v = \frac{1}{2}\langle v, Ax\rangle + \frac{1}{2}\langle x, Av\rangle = \langle v, Ax\rangle = Ax(v), \quad \forall\, x, v \in \mathscr{X},$$

so $Q'(x) = Ax \in \tilde{\mathscr{X}}$. By hypothesis, $\mathscr{X} = \mathscr{X}_0 \oplus \mathscr{K}$ as a direct sum of Banach spaces, where $\mathscr{K} := \operatorname{Ker} A$ and $\mathscr{X}_0 \subset \mathscr{X}$ is a closed subspace, and $\operatorname{Ran} A = \tilde{\mathscr{X}}$, so that $A \in \mathscr{L}(\mathscr{X}_0, \tilde{\mathscr{X}})$ is an isomorphism of Banach spaces by the Open Mapping Theorem. Note that for $x = z + \xi \in \mathscr{X}_0 \oplus \mathscr{K}$, we have

$$Q(z+\xi) = \frac{1}{2}\langle z+\xi, A(z+\xi)\rangle = \frac{1}{2}\langle z+\xi, Az\rangle = \frac{1}{2}\langle z, A(z+\xi)\rangle = \frac{1}{2}\langle z, Az\rangle = Q(z)^2,$$

while

$$Q'(z+\xi) = A(z+\xi) = Az = Q'(z).$$

Hence, it suffices to prove that the Łojasiewicz gradient inequality (3.1) holds for all $x \in \mathscr{X}_0$. For such $x \in \mathscr{X}_0$, we have

$$\|Q'(x)\|_{\tilde{\mathscr{X}}} = \|Ax\|_{\tilde{\mathscr{X}}} \geq \lambda \|x\|_{\mathscr{X}},$$

by writing $\|x\|_{\mathscr{X}} = \|A^{-1}Ax\|_{\mathscr{X}} \leq \|A^{-1}\|_{\mathscr{L}(\tilde{\mathscr{X}},\mathscr{X}_0)}\|Ax\|_{\tilde{\mathscr{X}}}$ and denoting $\lambda := \|A^{-1}\|_{\mathscr{L}(\tilde{\mathscr{X}},\mathscr{X}_0)} \in (0,\infty)$. On the other hand, for any $x \in \mathscr{X}$,

$$|Q(x)| \leq \frac{1}{2}|\langle x, Ax\rangle| \leq \frac{1}{2}\|x\|_{\mathscr{X}}\|Ax\|_{\mathscr{X}^*} \leq \frac{\kappa}{2}\|x\|_{\mathscr{X}}\|Ax\|_{\tilde{\mathscr{X}}} \leq \frac{\kappa\Lambda}{2}\|x\|_{\mathscr{X}}^2,$$

where we denote $\Lambda := \|A\|_{\mathscr{L}(\mathscr{X},\tilde{\mathscr{X}})} \in (0,\infty)$ and where κ is the norm of the continuous embedding, $\tilde{\mathscr{X}} \subset \mathscr{X}^*$. Therefore,

$$\|Q'(x)\|_{\tilde{\mathscr{X}}} \geq \lambda\|x\|_{\mathscr{X}} \geq \lambda\left(2|Q(x)|/\kappa\Lambda\right)^{1/2} = \lambda\sqrt{2/\kappa\Lambda}|Q(x)|^{1/2},$$

for all $x \in \mathscr{X}_0$ and this yields the Łojasiewicz gradient inequality (3.1) for all $x \in \mathscr{X}$. □

We have the following generalization of Lemma 1.4.

Lemma 3.2 (Łojasiewicz exponents and maps)**.** *Let $\mathscr{X}$, $\tilde{\mathscr{X}}$, and $\mathscr{Y}$ be Banach spaces over $\mathbb{K}$, and $\mathscr{V} \subset \mathscr{Y}$ and $\mathscr{U} \subset \mathscr{X}$ be open neighborhoods of the origins, and $\Phi : \mathscr{V} \to \mathscr{U}$ be an open C^1 map such that $\Phi(0) = 0$. Let $f : \mathscr{U} \to \mathbb{K}$ be a C^1 function such that $f(0) = 0$ and $f'(x) \in \tilde{\mathscr{X}}$ for all $x \in \mathscr{U}$. If $\Phi^*f(y)$ obeys the Łojasiewicz gradient inequality* (1.16) *with exponent $\theta \geq 0$ for all $y \in \mathscr{V}$ then, after possibly shrinking $\mathscr{U}$, the function $f(x)$ obeys the Łojasiewicz gradient inequality* (1.16) *with the same exponent θ and a possibly smaller constant $C \in (0,\infty)$, for all $x \in \mathscr{U}$.*

Proof. By hypothesis and (1.16) for the function $f \circ \Phi$ on $\mathscr{V}$, there is a constant $C \in (0,\infty)$ such that

$$\|(f \circ \Phi)'(y)\|_{\tilde{\mathscr{X}}} \geq C|(f \circ \Phi)(y)|^\theta, \quad \forall\, y \in \mathscr{V}.$$

Because Φ is an open map, $\Phi(\mathscr{V})$ is an open neighborhood of the origin in $\mathscr{X}$ and so by shrinking $\mathscr{U}$ if necessary, we may assume that $\Phi(\mathscr{V}) = \mathscr{U}$. Now $(f \circ \Phi)(y) = f(\Phi(y)) = f(x)$ for all $x \in \mathscr{U}$ and $y \in \Phi^{-1}(x)$ and therefore the preceding gradient inequality yields

$$\|(f \circ \Phi)'(y)\|_{\tilde{\mathscr{X}}} \geq C|f(x)|^\theta, \quad \forall\, x \in \mathscr{U} \text{ and } y \in \Phi^{-1}(x). \tag{3.2}$$

The Chain Rule yields

$$\begin{aligned}\|(f \circ \Phi)'(y)\|_{\tilde{\mathscr{X}}} &\leq \|f'(\Phi(y))\|_{\tilde{\mathscr{X}}}\|\Phi'(y)\|_{\mathscr{L}(\mathscr{X})}\\ &\leq M\|f'(\Phi(y))\|_{\tilde{\mathscr{X}}} \quad \forall\, y \in \mathscr{V},\end{aligned}$$

where $M := \sup_{y\in\mathscr{V}}\|\Phi'(y)\|_{\mathscr{L}(\mathscr{X})}$ and $M < \infty$ (possibly after shrinking $\mathscr{V}$). Because $\Phi(y) = x \in \mathscr{U}$, the preceding inequality simplifies to give

$$\|(f \circ \Phi)'(y)\|_{\tilde{\mathscr{X}}} \leq M\|f'(x)\|_{\tilde{\mathscr{X}}}, \quad \forall\, y \in \mathscr{V}. \tag{3.3}$$

By combining the inequalities (3.2) and (3.3), we obtain

$$\|f'(x)\|_{\tilde{\mathscr{X}}} \geq (C/M)|f(x)|^\theta, \quad \forall\, x \in \mathscr{U},$$

which is (1.16) with constant C/M, as desired. □

Proof of Theorem 7. By hypothesis, f is a C^{p+1} Morse–Bott function at the origin and so, possibly after shrinking $\mathscr{U}$, Theorem 2.14 provides an open neighborhood, $\mathscr{V}$, of the origin in $\mathscr{X}$ and a C^p diffeomorphism, $\Phi : \mathscr{V} \to \mathscr{U}$, such that $\Phi(0) = 0$ and

$$f \circ \Phi(y) = \langle y, Ay\rangle, \quad \forall\, y \in \mathscr{V},$$

where $A = f''(0) = (f \circ \Phi)''(0) \in \mathscr{L}_{\mathrm{sym}}(\mathscr{X}, \tilde{\mathscr{X}})$. By Definition 1.8 (1), the kernel of A has a closed complement in $\mathscr{X}$ and the range of A is $\tilde{\mathscr{X}}$. Lemma 3.1 then asserts that the quadratic function, $Q(y) = \langle y, Ay\rangle$ for all $y \in \mathscr{X}$, has Łojasiewicz exponent $1/2$, while Lemma 3.2 implies that the functions f and $f \circ \Phi$ have the same Łojasiewicz exponent, namely $1/2$. This completes the proof of Theorem 7. □

3.2. **Łojasiewicz gradient inequality for analytic functions.** In this subsection, we apply Theorem 5 to prove Theorem 9, and hence Theorem 8 upon choosing $\tilde{\mathscr{X}} = \mathscr{X}^*$. We first have the elementary

Lemma 3.3 (Invariance of Łojasiewicz exponent under direct sum addition or subtraction of a quadratic form)**.** *Let $\mathscr{X}$, $\tilde{\mathscr{X}}$, $\mathscr{Y}$, and $\tilde{\mathscr{Y}}$ be Banach spaces over $\mathbb{K}$ with continuous embeddings, $\tilde{\mathscr{X}} \subset \mathscr{X}^*$ and $\tilde{\mathscr{V}} \subset \mathscr{Y}^*$, and $\theta \in [1/2, 1)$ be a constant, $\mathscr{U} \subset \mathscr{X}$ be an open neighborhood of the origin, $f : \mathscr{X} \supset \mathscr{U} \to \mathbb{K}$ be a C^2 function with $f(0) = 0 \in \mathbb{K}$ and $f'(0) = 0$ and $f'(x) \in \tilde{\mathscr{X}}$ for all $x \in \mathscr{U}$, and $Q : \mathscr{Y} \ni y \mapsto Q(y) = \frac{1}{2}\langle y, Ay\rangle \in \mathbb{K}$ be defined by an operator, $A \in \mathscr{L}_{\mathrm{sym}}(\mathscr{Y}, \tilde{\mathscr{Y}})$, whose kernel is complemented in $\mathscr{Y}$ and whose range is $\tilde{\mathscr{Y}}$. If $f_Q : \mathscr{U} \times \mathscr{Y} \to \mathbb{K}$ is a C^2 function defined by $f_Q(x, y) := f(x)+Q(y)$ for $(x, y) \in \mathscr{U} \times \mathscr{Y}$, then there are constants $C, C_0 \in (0, \infty)$ and an open neighborhood $\mathscr{V} \subset \mathscr{Y}$ of the origin such that, after possibly shrinking $\mathscr{U}$, the following holds: f has Łojasiewicz exponent θ on $\mathscr{U}$, that is,*

$$\|f'(x)\|_{\tilde{\mathscr{X}}} \geq C_0|f(x)|^\theta, \quad \forall\, x \in \mathscr{U}, \tag{3.4}$$

if and only if f_Q has Łojasiewicz exponent θ on $\mathscr{U} \times \mathscr{V}$, that is,

$$\|f_Q'(x, y)\|_{\tilde{\mathscr{X}} \oplus \tilde{\mathscr{Y}}} \geq C|f_Q(x, y)|^\theta, \quad \forall\, (x, y) \in \mathscr{U} \times \mathscr{V}. \tag{3.5}$$

Proof. Let $\alpha := 1/\theta \in (1, 2]$ and suppose that Inequality (3.4) holds. Since $f'(0) = 0$ and $Q'(0) = 0$, we may assume $\|f'(x) \oplus Q'(y)\|_{\tilde{\mathscr{X}} \oplus \tilde{\mathscr{Y}}} \leq 1$ for all $(x, y) \in \mathscr{U} \times \mathscr{V}$, for small enough $\mathscr{V}$ and after possibly shrinking $\mathscr{U}$. Observe that for all $(x, y) \in \mathscr{U} \times \mathscr{V}$,

$$\begin{aligned}
|f_Q(x, y)| &\leq |f(x)| + |Q(y)| \\
&\leq C_0\|f'(x)\|^\alpha_{\tilde{\mathscr{X}}} + C_1\|Q'(y)\|^2_{\tilde{\mathscr{Y}}} \quad \text{(by Lemma 3.1 and Inequality (3.4))} \\
&\leq C\left(\|f'(x)\|^\alpha_{\tilde{\mathscr{X}}} + \|Q'(y)\|^2_{\tilde{\mathscr{Y}}}\right) \\
&\leq C\left(\|f'(x) \oplus Q'(y)\|^\alpha_{\tilde{\mathscr{X}} \oplus \tilde{\mathscr{Y}}} + \|f'(x) \oplus Q'(y)\|^2_{\tilde{\mathscr{X}} \oplus \tilde{\mathscr{Y}}}\right) \\
&\leq C\|f'(x) \oplus Q'(y)\|^\alpha_{\tilde{\mathscr{X}} \oplus \tilde{\mathscr{Y}}} \quad \text{(as } \|f'(x) \oplus Q'(y)\|_{\tilde{\mathscr{X}} \oplus \tilde{\mathscr{Y}}} \leq 1 \text{ and } \alpha \in (1, 2]) \\
&= C\|f_Q'(x, y)\|^\alpha_{\tilde{\mathscr{X}} \oplus \tilde{\mathscr{Y}}},
\end{aligned}$$

where $C = \max\{C_0, C_1\}$ and $f_Q'(x, y) = f'(x) \oplus Q'(y)$. Taking the $1/\alpha$ root of the preceding inequality yields Inequality (3.5).

Conversely, suppose that Inequality (3.5) holds. For all $x \in \mathscr{U}$,

$$\begin{aligned}
|f(x)| &= |f_Q(x, 0)| \\
&\leq C\|f_Q'(x, 0)\|^\alpha_{\tilde{\mathscr{X}} \oplus \tilde{\mathscr{Y}}} \quad \text{(by Inequality (3.5))} \\
&= C\|f'(x) \oplus Q'(0)\|^\alpha_{\tilde{\mathscr{X}} \oplus \tilde{\mathscr{Y}}} \\
&= C\left(\|f'(x)\|_{\tilde{\mathscr{X}}} + \|Q'(0)\|_{\tilde{\mathscr{Y}}}\right)^\alpha \\
&= C\|f'(x)\|^\alpha_{\tilde{\mathscr{X}}},
\end{aligned}$$

which gives Inequality (3.4) after taking the $1/\alpha$ root. This completes the proof of Lemma 3.3. □

We can now give the

Proof of Theorem 9. The operator $f''(0) \in \mathscr{L}_{\mathrm{sym}}(\mathscr{X}, \tilde{\mathscr{X}})$ is Fredholm by hypothesis, with finite-dimensional kernel, $\mathscr{K} := \operatorname{Ker} f''(0)$, and closed complement, $\mathscr{X}_0 \subset \mathscr{X}$, such that $\mathscr{X} = \mathscr{X}_0 \oplus \mathscr{K}$. Similarly, let $\tilde{\mathscr{X}}_0 := \operatorname{Ran} f''(0) \subset \tilde{\mathscr{X}}$ denote the closed range of $f''(0)$ with finite-dimensional complement, $\tilde{\mathscr{K}} \cong \mathscr{K} = \operatorname{Ker} f''(0)$, and $\tilde{\mathscr{X}}_0 \cong \mathscr{X}_0$ (see Lemma 2.4). Therefore, writing $x = (w, \xi) \in \mathscr{X} = \mathscr{X}_0 \oplus \mathscr{K}$,

$$f''(0,0) = \begin{pmatrix} A_0 & 0 \\ 0 & 0 \end{pmatrix} : \mathscr{X}_0 \oplus \mathscr{K} \to \tilde{\mathscr{X}}_0 \oplus \tilde{\mathscr{K}},$$

where $A_0 = D_w^2 f(0,0) \in \mathscr{L}(\mathscr{X}_0, \tilde{\mathscr{X}}_0)$ is symmetric with respect to the continuous embedding, $\tilde{\mathscr{X}}_0 \subset \mathscr{X}_0^*$, and canonical pairing, $\mathscr{X}_0 \times \mathscr{X}_0^* \to \mathbb{K}$. Moreover, A_0 is bijective and continuous by construction, so it is invertible by the Open Mapping Theorem.

By hypothesis, f is analytic and so, possibly after shrinking $\mathscr{U}$, Theorem 5 provides an open neighborhood $\mathscr{V}$ of the origin in $\mathscr{X}$ and an analytic diffeomorphism, $\Phi : \mathscr{V} \to \mathscr{U}$, such that $\Phi(0,0) = (0,0)$ and

$$f \circ \Phi(z, \xi) = g(\xi) + \langle z, A_0 z \rangle, \quad \forall\, y = (z, \xi) \in \mathscr{V},$$

where $A_0 = D_z^2(f \circ \Phi)(0,0) \in \mathscr{L}_{\mathrm{sym}}(\mathscr{X}_0, \tilde{\mathscr{X}}_0)$ and $g(\xi) := f(\Phi(0,\xi))$ for all ξ in V, an open neighborhood of the origin in $\mathscr{K}$ defined as the image of the projection of $\mathscr{V} \subset \mathscr{X} = \mathscr{X}_0 \oplus \mathscr{K}$ onto the factor $\mathscr{K}$. Lemma 3.1 then asserts that the quadratic function, $Q(z) := \langle z, A_0 z \rangle$ for $z \in \mathscr{X}_0$, has Łojasiewicz exponent $1/2$ on an open neighborhood of the origin, while Lemmas 3.2 and 3.3 imply that the functions $f : \mathscr{U} \to \mathbb{K}$ and $f \circ \Phi : \mathscr{V} \to \mathbb{K}$ and $g : V \to \mathbb{K}$ have the same Łojasiewicz exponent. But $\mathscr{K}$ is a finite-dimensional vector space over $\mathbb{K}$ and g is analytic and thus obeys the classical Łojasiewicz gradient inequality for some exponent $\theta \in [1/2, 1)$ by Theorem 1.1. This completes the proof of Theorem 9. □

4. Analytic functions with Łojasiewicz exponent one half are Morse–Bott

Our goal in this section is to complete the proof of Theorem 2 and hence Theorem 1 upon choosing $\tilde{\mathscr{X}} = \mathscr{X}^*$.

Proof of Theorem 2. Recall that $f''(0) \in \mathscr{L}_{\mathrm{sym}}(\mathscr{X}, \tilde{\mathscr{X}})$ is a Fredholm operator by hypothesis. Let $\mathscr{K} := \operatorname{Ker} f''(0) \subset \mathscr{X}$ denote the finite-dimensional kernel with closed complement $\mathscr{X}_0$, so $\mathscr{X} = \mathscr{X}_0 \oplus \mathscr{K}$, and let $\tilde{\mathscr{X}}_0 := \operatorname{Ran} f''(0) \subset \tilde{\mathscr{X}}$ denote the closed range, with finite-dimensional complement $\tilde{\mathscr{K}}$, so $\tilde{\mathscr{X}} = \tilde{\mathscr{X}}_0 \oplus \tilde{\mathscr{K}}$.

We apply the Morse Lemma for functions on Banach spaces with degenerate critical points (Theorem 5) to f to produce — after possibly shrinking the open neighborhood $\mathscr{U}$ of the origin in $\mathscr{X}$ — an analytic diffeomorphism, Φ, from an open neighborhood $\mathscr{V}$ of the origin in $\mathscr{X} = \mathscr{X}_0 \oplus \mathscr{K}$ onto $\mathscr{U}$ such that

$$\Phi^* f(z, \xi) = \frac{1}{2}\langle z, A_0 z \rangle + \Phi^* f(0, \xi), \quad \forall\, y = (z, \xi) \in \mathscr{V} \subset \mathscr{X}_0 \oplus \mathscr{K},$$

where $A_0 := D_z^2 \Phi^* f(0,0) \in \mathscr{L}_{\mathrm{sym}}(\mathscr{X}_0, \tilde{\mathscr{X}}_0)$. Again, the operator A_0 is bijective and continuous by construction, so it is invertible by the Open Mapping Theorem. The function $g := f(\Phi(0, \cdot))$ is analytic on an open neighborhood V of the origin in $\mathscr{K}$ defined as the image of the projection of $\mathscr{V} \subset \mathscr{X} = \mathscr{X}_0 \oplus \mathscr{K}$ onto the factor $\mathscr{K}$. By construction, $g(0) = 0$ and $g'(0) = 0$.

By shrinking $\mathscr{V}$ if necessary, we may assume without loss of generality that V is connected. If g is identically zero on V, then we are done. Otherwise, if g is not identically zero on V, our hypothesis that f has Łojasiewicz exponent $1/2$ and Lemmas 3.2 and 3.3 imply that g has Łojasiewicz exponent $1/2$ as well.

If $\operatorname{Ker} g''(0) = \{0\}$, then $\operatorname{Coker} g''(0) = \{0\}$, since $g''(0) \in \mathscr{L}(\mathscr{K}, \mathscr{K}^*)$ is symmetric[13], and thus $g''(0) \in \mathscr{L}(\mathscr{K}, \mathscr{K}^*)$ is invertible. Hence, g is a Morse–Bott function — in fact a Morse function with $\operatorname{Crit} g = \{0\}$ — and thus $\Phi^* f$ is a Morse–Bott function. But then f itself must be a Morse-Bott function since Φ is a diffeomorphism from one open neighborhood of the origin in $\mathscr{X}$ onto another and this would complete the proof of Theorem 2.

If $\operatorname{Ker} g''(0) \neq \{0\}$, then there exists $v \in \mathscr{K}$ such that $\|v\|_{\mathscr{K}} = 1$ and, since g is analytic, an integer $m \geq 3$ such that $g^{(m)}(0)v^m \neq 0$. The Taylor Formula then yields

$$g(tv) = \frac{1}{m!} g^{(m)}(0)v^m\, t^m + \frac{1}{(m+1)!} \int_0^t g^{(m+1)}(sv)v^{m+1} s^m \, ds,$$
$$g'(tv) = \frac{1}{(m-1)!} g^{(m)}(0)v^{m-1}\, t^{m-1} + \frac{1}{m!} \int_0^t g^{(m+1)}(sv)v^{m} s^{m-1} \, ds,$$

for all $t \in \mathbb{K}$ such that $tv \in V$. Therefore, after possibly further shrinking $\mathscr{V}$ and hence V,

$$\frac{1}{C}|\xi|^{(m-1)/m} \leq \|g'(\xi)\|_{\mathscr{K}^*} \leq C|\xi|^{(m-1)/m}, \quad \forall\, \xi \in V \cap \mathbb{K}v,$$

for a positive constant C depending at most on m and $|g^{(m)}(0)v^m|$ and $\sup_{\zeta \in V} \|g^{(m+1)}(\zeta)\|$, where $g^{(m+1)}(\zeta) \in \otimes^{m+1} \mathscr{K}^*$. But $(m-1)/m \geq 2/3$ and the inequality,

$$\|g'(\xi)\|_{\mathscr{K}^*} \leq C|\xi|^{(m-1)/m}, \quad \forall\, \xi \in V \cap \mathbb{K}v,$$

contradicts the fact that g has Łojasiewicz exponent $1/2$, since (1.1) would yield, after possibly further shrinking V,

$$\|g'(\xi)\|_{\mathscr{K}^*} \geq C_0 |g(\xi)|^{1/2}, \quad \forall\, \xi \in V,$$

for some positive constant C_0. Hence, $\operatorname{Ker} g''(0) = \{0\}$, completing the proof of Theorem 2. □

Appendix A. Rate of convergence of a gradient flow for a function obeying a Łojasiewicz gradient inequality

We recall the following enhancement of Huang [47, Theorem 3.4.8].

Theorem A.1 (Convergence rate under the validity of a Łojasiewicz gradient inequality)**.** *(See Feehan [29, Theorem 3].) Let $\mathscr{U}$ be an open subset of a real Banach space, $\mathscr{X}$, that is continuously embedded and dense in a Hilbert space, $\mathscr{H}$. Let $\mathscr{E} : \mathscr{U} \subset \mathscr{X} \to \mathbb{R}$ be an analytic function with gradient map $\mathscr{E}' : \mathscr{U} \subset \mathscr{X} \to \mathscr{H}$ and $x_\infty \in \mathscr{U}$ be a critical point, that is, $\mathscr{E}'(x_\infty) = 0$. Assume that there are constants, $c \in (0, \infty)$, and $\sigma \in (0, 1]$, and $\theta \in [1/2, 1)$ such that*

$$\|\mathscr{E}'(x)\|_{\mathscr{H}} \geq c|\mathscr{E}(x) - \mathscr{E}(x_\infty)|^\theta, \quad \forall\, x \in \mathscr{U}_\sigma, \tag{A.1}$$

where $\mathscr{U}_\sigma := \{x \in \mathscr{X} : \|x - x_\infty\|_{\mathscr{X}} < \sigma\}$. Let $u \in C^\infty([0,\infty); \mathscr{X})$ be a solution to the gradient system,

$$\dot{u}(t) = -\mathscr{E}'(u(t)), \quad t \in (0, \infty), \tag{A.2}$$

and assume that the orbit $O(u) := \{u(t) : t \geq 0\} \subset \mathscr{X}$ obeys $O(u) \subset \mathscr{U}_\sigma$. Then there exists $u_\infty \in \mathscr{H}$ such that

$$\|u(t) - u_\infty\|_{\mathscr{H}} \leq \Psi(t), \quad t \geq 0, \tag{A.3}$$

[13] Note that $\mathscr{K}$ is finite-dimensional.

where

$$\Psi(t) := \begin{cases} \dfrac{1}{c(1-\theta)}\left(c^2(2\theta-1)t + (\gamma-a)^{1-2\theta}\right)^{-(1-\theta)/(2\theta-1)}, & 1/2 < \theta < 1, \\ \dfrac{2}{c}\sqrt{\gamma-a}\exp(-c^2t/2), & \theta = 1/2, \end{cases} \tag{A.4}$$

and a, γ are constants such that $\gamma > a$ and

$$a \leq \mathscr{E}(v) \leq \gamma, \quad \forall\, v \in \mathscr{U}.$$

If in addition u obeys Hypothesis A.2, then $u_\infty \in \mathscr{X}$ and

$$\|u(t+1) - u_\infty\|_{\mathscr{X}} \leq 2C_1\Psi(t), \quad t \geq 0, \tag{A.5}$$

where $C_1 \in [1,\infty)$ is the constant in Hypothesis A.2 for $\delta = 1$.

We recall the

Hypothesis A.2 (*A priori* interior estimate for a trajectory)**.** (See Feehan [29, Hypothesis 2.1].) Let $\mathscr{X}$ be a Banach space that is continuously embedded in a Hilbert space $\mathscr{H}$. If $\delta \in (0,\infty)$ is a constant, then there is a constant $C_1 = C_1(\delta) \in [1,\infty)$ with the following significance. If $S, T \in \mathbb{R}$ are constants obeying $S + \delta \leq T$ and $u \in C^\infty([S,T);\mathscr{X})$, we say that $\dot{u} \in C^\infty([S,T);\mathscr{X})$ obeys an *a priori interior estimate on* $(0,T]$ if

$$\int_{S+\delta}^T \|\dot{u}(t)\|_{\mathscr{X}}\,dt \leq C_1 \int_S^T \|\dot{u}(t)\|_{\mathscr{H}}\,dt. \tag{A.6}$$

In applications, $u \in C^\infty([S,T);\mathscr{X})$ in Hypothesis A.2 will often be a solution to a quasi-linear parabolic partial differential system, from which an *a priori* estimate (A.6) may be deduced. For example, Hypothesis A.2 is verified by Feehan [29, Lemma 17.12] for a nonlinear evolution equation on a Banach space $\mathcal{V}$ of the form (see Caps [18], Henry [41], Pazy [67], Sell and You [75], Tanabe [82, 83] or Yagi [91])

$$\frac{du}{dt} + \mathcal{A}u = \mathcal{F}(t,u(t)), \quad t \geq 0, \quad u(0) = u_0, \tag{A.7}$$

where $\mathcal{A}$ is a positive, sectorial, unbounded operator on a Banach space, $\mathcal{W}$, with domain $\mathcal{V}^2 \subset \mathcal{W}$ and the nonlinearity, $\mathcal{F}$, has suitable properties.

Results on the rate of convergence of a gradient flow defined by a function obeying a Łojasiewicz gradient inequality in specific examples have been proved earlier — see Simon [76] and Adams and Simon [2] for a restricted class of analytic energy functions arising in geometric analysis and Rade [72, Proposition 7.4] for the Yang-Mills energy function on connections on principal bundles over a closed smooth manifold of dimension two or three. For a recent example, see Carlotto, Chodosh, and Rubinstein [19, Theorem 1] for the Yamabe function on Riemannian metrics over closed smooth manifolds of dimension greater than or equal to three.

Appendix B. Morse–Bott functions and quadratic simple normal crossing functions

We are often asked about the relationship between Morse–Bott functions and quadratic simple normal crossing functions as in (1.5), so we explain the relationship in this section for $\mathbb{K} = \mathbb{R}$; the analogous discussion applies for $\mathbb{K} = \mathbb{C}$.

For an integer $p \geq 1$ and writing $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$, we let $\mathbb{RP}^{p-1} = (\mathbb{R}^p \setminus \{0\})/\mathbb{R}^* = S^{p-1}/\{\pm 1\}$ denote real projective space, so $\mathbb{RP}^0 \cong \{1\}$ and $\mathbb{RP}^1 \cong S^1$ while $\mathbb{RP}^{p-1}$ with $p \geq 3$ is obtained by identifying antipodal points of the sphere, S^{p-1}.

Definition B.1 (Blowup at a point and exceptional divisor)**.** (See Krantz and Parks [49, Definition 6.2.2].) Let $p \geq 1$ be an integer and W be an open neighborhood of the origin in $\mathbb{R}^p$. The *blowup* of W at the origin is the set

$$\widetilde{W} := \{(x, \ell) \in W \times \mathbb{R}^{p-1} : x \in \ell\},$$

where $\pi : \widetilde{W} \ni (x, \ell) \mapsto x \in W$ is the *blowup map* and $E := \pi^{-1}(0) \subset \widetilde{W}$ is the *exceptional divisor*.

The set $\widetilde{W}$ is a real analytic manifold and the quotient map $\pi : \widetilde{W} \to W$ is real analytic and restricts to a real analytic diffeomorphism, $\pi : \widetilde{W} \setminus E \cong W \setminus \{0\}$. By viewing $\mathbb{R}^{p-1} = S^{p-1}/\{\pm 1\}$ and $\mathbb{R}^p \setminus \{0\} = \mathbb{R}_+ \times S^{p-1}$, we may also write

$$\begin{aligned}
\widetilde{W} &= \{(x, \ell) \in W \times \mathbb{R}^{p-1} : x \in \ell\} \\
&= \{(x, [u]) \in W \times S^{p-1}/\{\pm 1\} : x \in \mathbb{R}u\} \\
&= \{(x, [u]) \in W \times S^{p-1}/\{\pm 1\} : x = \pm|x|u\} \\
&= \{(x, u) \in W \times S^{p-1} : (x, u) \sim (y, v) \text{ if } |x| = |y| \text{ and } u = \pm v\} \\
&= \{(s, u) \in \mathbb{R} \times S^{p-1} : su \in W \text{ and } (s, u) \sim (t, v) \text{ if } (t, v) = \pm(s, u)\}
\end{aligned}$$

where $[u] = \{\pm u\}$ and, in the last writing, the blowup map is $\pi : \widetilde{W} \ni [s, u] \mapsto su \in W$ and $\pi^{-1}(0) = \{\{0\} \times S^{p-1}\}/\{\pm 1\} \cong S^{p-1}/\{\pm 1\} = \mathbb{R}^{p-1}$ is the exceptional divisor.

If W is an open neighborhood of the origin in $\mathbb{R}^p$ or the half-space $\mathbb{H}^p = \{x \in \mathbb{R}^p : x_p \geq 0\}$, then we could alternatively define the blowup of W at the origin to be the real analytic manifold with boundary,

$$\widehat{W} := \{(r, u) \in [0, \infty) \times S^{p-1} : ru \in W\},$$

following the usual definition of polar coordinates on $\mathbb{R}^p \setminus \{0\}$. The map $\pi : \widehat{W} \ni (r, u) \mapsto ru \in W$ is the blowup map and $\pi^{-1}(0) = \{0\} \times S^{p-1} \cong S^{p-1}$ is now the exceptional divisor.

Suppose now that $U \subset \mathbb{R}^d$ is an open neighborhood of the origin and $f : U \to \mathbb{R}$ is a C^2 function with $f(0) = 0$ and $f'(0) = 0$ and that is Morse–Bott at the origin in the sense of Definition 1.5 (1). Thus, after possibly shrinking U, we have that $\operatorname{Crit} f$ is a C^2 submanifold of U of dimension $c = \dim \operatorname{Ker} f''(0)$. Moreover, we may further assume that U is connected and so $\operatorname{Crit} f \subset f^{-1}(0)$.

Theorem 2.10 and Remark 2.12 (the Morse–Bott Lemma) imply, after possibly shrinking U, that one can find an neighborhood V of the origin in $\mathbb{R}^d$ and a C^2 diffeomorphism[14], $\Phi : \mathbb{R}^d \supset V \ni y \mapsto x \in U \subset \mathbb{R}^d$, such that $\Phi(0) = 0$ and

$$f \circ \Phi(y) = \sum_{i=1}^{p} y_i^2 - \sum_{i=p+1}^{p+n} y_i^2, \quad \forall\, y \in V.$$

Note that $p + n = d - c$ and

$$\operatorname{Crit} f \circ \Phi = V \cap \bigcap_{i=1}^{d-c} \{y_i = 0\}.$$

If $n = 0$ and thus $1 \leq p = d - c$, we may write $(y_1, \dots, y_p) = su$, for $s \in [0, \infty)$ and $u \in S^{p-1} \subset \mathbb{R}^p$, so that

$$f \circ \varpi(s, u, y_{p+1}, \cdots, y_d) = s^2, \quad \forall\, (su, y_{p+1}, \cdots, y_d) \in U,$$

[14]While for clarity we have restricted our attention in this article to functions f which are C^{p+2} with $p \geq 1$, the Morse–Bott Lemma holds for C^2 functions on Euclidean space: see Banyaga and Hurtubise [8, Theorem 2].

where we define $\varpi(s,u,y_{p+1},\cdots,y_d) := \Phi(su,y_{p+1},\cdots,y_d)$. We see that ϖ gives a C^2 map from an open neighborhood V of the origin in $[0,\infty)\times S^{p-1}\times\mathbb{R}^{d-p}$ onto $U\subset\mathbb{R}^d$ such that $\varpi(0)=0$ and

$$\varpi(\{s=0\}\cap V) = U\cap\bigcap_{i=1}^{p}\{y_i=0\} = U\cap\bigcap_{i=1}^{d-c}\{y_i=0\},$$

and ϖ is a diffeomorphism from $V\setminus\{s=0\}$ onto its image.

Similarly, if $p=0$ and thus $1\le n=d-c$, we may write $(y_1,\dots,y_n)=tv$, for $t\in[0,\infty)$ and $v\in S^{n-1}\subset\mathbb{R}^n$, so that

$$f\circ\varpi(t,v,y_{n+1},\cdots,y_d) = -t^2,\quad \forall\,(tv,y_{n+1},\cdots,y_d)\in U,$$

where we define $\varpi(t,v,y_{n+1},\cdots,y_d)=\Phi(tv,y_{n+1},\cdots,y_d)$. We see that ϖ gives a C^2 map from an open neighborhood of the origin in $[0,\infty)\times S^{p-1}\times\mathbb{R}^{d-p}$ into $\mathbb{R}^d$ such that $\varpi(0)=0$ and

$$\varpi(\{t=0\}\cap V) = U\cap\bigcap_{i=1}^{n}\{y_i=0\} = U\cap\bigcap_{i=1}^{d-c}\{y_i=0\},$$

and ϖ is a diffeomorphism from $V\setminus\{t=0\}$ onto its image.

Finally, if $n\ge 1$ and $p\ge 1$, we may write $(y_1,\dots,y_p)=su$ and $(y_{p+1},\dots,y_{p+n})=tv$, for $s,t\in[0,\infty)$ and $u\in S^{p-1}$ and $v\in S^{n-1}$, so that

$$f\circ\varpi(s,t,u,v,y_{p+n+1},\cdots,y_d) = s^2-t^2,\quad \forall\,(su,tv,y_{p+n+1},\cdots,y_d)\in U,$$

where we define $\varpi(s,t,u,v,y_{p+n+1},\cdots,y_d) := \Phi(su,tv,y_{p+n+1},\cdots,y_d)$. We see that ϖ gives a C^2 map from an open neighborhood of the origin in $[0,\infty)\times[0,\infty)\times S^{p-1}\times S^{n-1}\times\mathbb{R}^{d-n-p}$ into $\mathbb{R}^d$ such that $\varpi(0)=0$ and

$$\varpi(\{s=0\}\cap\{t=0\}\cap W) = V\cap\bigcap_{i=1}^{n+p}\{y_i=0\} = V\cap\bigcap_{i=1}^{d-c}\{y_i=0\},$$

after possibly shrinking V and ϖ is a diffeomorphism from $W\setminus(\{s=0\}\cup\{t=0\})$ onto its image.

Define a diffeomorphism of $\mathbb{R}^2$ by $(t_1,t_2)\mapsto(s,t)=\varphi(t_1,t_2)$ where $t_1=s+t$ and $t_2=s-t$, so that $s=\frac{1}{2}(t_1+t_2)$ and $t=\frac{1}{2}(t_1-t_2)$. Hence, we obtain

$$f\circ\Pi(t_1,t_2,u,v,y_{p+n+1},\cdots,y_d) = t_1t_2,\quad \forall\,(\varphi_1(t_1,t_2)u,\varphi_2(t_1,t_2)v,y_{p+n+1},\cdots,y_d)\in U,$$

where we define $\Pi(t_1,t_2,u,v,y_{p+n+1},\cdots,y_d) := \Phi(\varphi_1(t_1,t_2)u,\varphi_2(t_1,t_2)v,y_{p+n+1},\cdots,y_d)$. We see that Π gives a C^2 map from an open neighborhood of the origin in $\{(t_1,t_2)\in[0,\infty)\times\mathbb{R} : |t_2|\le t_1\}\times S^{p-1}\times S^{n-1}\times\mathbb{R}^{d-n-p}$ into $\mathbb{R}^d$ such that $\Pi(0)=0$ and

$$\Pi(\{t_1=0\}\cap\{t_2=0\}\cap V) = U\cap\bigcap_{i=1}^{d-c}\{y_i=0\},$$

and Π is a diffeomorphism from $V\setminus(\{t_1=0\}\cup\{t_2=0\})$ onto its image.

In the preceding discussion we could have replaced the roles of the blowups $[0,\infty)\times S^{p-1}$ or $[0,\infty)\times S^{n-1}$ by $(\mathbb{R}\times S^{p-1})/\{\pm1\}$ or $(\mathbb{R}\times S^{n-1})/\{\pm1\}$ and the roles of the exceptional divisors, S^{p-1} or S^{n-1} by $\mathbb{RP}^{p-1}$ or $\mathbb{RP}^{n-1}$, the only difference being an increase in notational complexity. In summary, we have proved the

Proposition B.2 (Pull-back of a Morse–Bott function to a quadratic simple normal crossing function). *Let $d\ge 2$ be an integer, $U\subset\mathbb{R}^d$ be an open neighborhood of the origin, and $f:U\to\mathbb{R}$ be a C^2 function that is Morse–Bott at the origin and obeys $f(0)=0$. Then, after possibly shrinking U, there are an open neighborhood V of the origin in $\mathbb{R}^d$ and a C^2 map, $\pi:V\to U$,* *such that* π *restricts to a diffeomorphism from* $V \setminus \{y_1 = 0\}$ *or* $V \setminus (\{y_1 = 0\} \cup \{y_2 = 0\})$ *onto its image and*

$$\pi^* f(y) = \pm y_1^2 \quad or \quad y_1 y_2, \quad \forall\, y = (y_1, \ldots, y_d) \in V,$$

and $\pi(\operatorname{Crit} f \circ \pi) = \operatorname{Crit} f$, *where* $\operatorname{Crit} f \circ \pi = \{y_1 = 0\} \cap V$ *or* $(\{y_1 = y_2 = 0\}) \cap V$.

Appendix C. Integrability and the Morse-Bott condition for the harmonic map energy function

Following Lemaire and Wood [54, Section 1], we review the concept of *integrability* of a *Jacobi field* along a harmonic map, describe the relation between integrability and the Morse-Bott condition for the harmonic map energy function at a harmonic map. We then indicate some of the few examples where integrability is known for harmonic maps.[15]

We begin by recalling the *second variation of the energy.* For a smooth two-parameter variation, $f_{t,s} : M \to N$, of a map $f : M \to N$ with $\partial f_{t,s}/\partial t|_{(0,0)} = v$ and $\partial^2 f_{t,s}/\partial s|_{(0,0)} = w$, the *Hessian* of f is defined by

$$\operatorname{Hess}_f(v, w) := \left.\frac{\partial^2 \mathscr{E}(f_{t,s})}{\partial t \partial s}\right|_{(0,0)}.$$

One has

$$\operatorname{Hess}_f(v, w) = (J_f(v), w)_{L^2(M,g)},$$

where

$$J_f(v) := \Delta v - \operatorname{tr} R^N(df, v) df$$

is called the *Jacobi operator*, a self-adjoint linear elliptic differential operator. Here, Δ denotes the Laplacian induced on $f^{-1}TN$ and the sign conventions on Δ and the curvature R^N are those of Eells and Lemaire [27].

Let v be a *vector field along* f, that is, a smooth section of $f^{-1}TN$, where $f : M \to N$ is a smooth map. Then v is called a *Jacobi field* (for the energy) if $J_f(v) = 0$. The space of Jacobi fields, $\operatorname{Ker} J_f$, is finite-dimensional and its dimension is called the $(\mathscr{E})$-*nullity* of f.

Definition C.1 (Integrability of a Jacobi field along a harmonic map)**.** [54, Definition 1.2] A Jacobi field v along a harmonic map, $f : M \to N$, is said to be *integrable* if there is a smooth family of harmonic maps, $f_t : M \to N$ for $t \in (-\varepsilon, \varepsilon)$, such that $f_0 = f$ and $v = \partial f_t/\partial t|_{t=0}$.

Kwon proved the following alternative characterization of the integrability condition in Definition C.1, based on results of Simon [77, pp. 270–272] and Adams and Simon [2, Lemma 1].

Proposition C.2. *(See Kwon* [52, Definition 4.4 and Proposition 4.1]*.) Let* $\varphi_0 : (M, g) \to (N, h)$ *be a harmonic map between real-analytic Riemannian manifolds. Then all Jacobi fields along* φ_0 *are integrable if and only if the space of harmonic maps* ($C^{2,\alpha}$-*) close to* φ_0 *is a smooth manifold, whose tangent space at* φ_0 *is* $\operatorname{Ker} \mathscr{E}''(\varphi_0)$.

It follows that for two real-analytic manifolds, all Jacobi fields along all harmonic maps are integrable if and only if the space of harmonic maps is a manifold whose tangent bundle is given by the Jacobi fields [54, p. 470]. By Definition 1.5, the conclusion of Proposition C.2 is equivalent to the assertion that all Jacobi fields along φ_0 are integrable if and only if the harmonic map energy function $\mathscr{E}$ is Morse-Bott at φ_0.

For a further discussion of integrability and additional references, see Adams and Simon [2, Section 1], Kwon [52, Section 4.1], and Simon [77, pp. 270–272].

[15] This appendix is quoted in full from Feehan and Maridakis [33, Appendix A] and further edited.

According to [54, Theorem 1.3] any Jacobi field along a harmonic map from S^2 to $\mathbb{CP}^2$ is integrable, where the two-sphere S^2 has its unique conformal structure and the complex projective space $\mathbb{CP}^2$ has its standard Fubini-Study metric of holomorphic sectional curvature 1; see Crawford [24] for additional results.

From the list of examples provided by Lemaire and Wood [54, p. 471], there are few other examples of families of harmonic maps that are guaranteed to be integrable, with the list including harmonic maps from S^2 to S^2 but excluding harmonic maps from S^2 to S^3 or S^4 [55].

Fernández [34] has proved that the space $\mathrm{Harm}_d(S^2, S^{2n})$ of degree-d harmonic maps from S^2 into S^{2n} has dimension $2d+n^2$. However, thus far, integrability for such maps is known only when $n = 1$. Bolton and Fernandez [12] provide a nice survey of what is known regarding regularity of $\mathrm{Harm}_d(S^2, S^{2n})$: they recall that $\mathrm{Harm}_d(S^2, S^2)$ is known to be a smooth manifold, outline a proof that $\mathrm{Harm}_d(S^2, S^6)$ is also a smooth manifold, and survey results on the structure of $\mathrm{Harm}_d(S^2, S^4)$ and why that space is not a smooth manifold.

References

[1] R. Abraham, J. E. Marsden, and T. Ratiu, *Manifolds, tensor analysis, and applications*, second ed., Springer, New York, 1988. MR 960687 (89f:58001)

[2] D. Adams and L. Simon, *Rates of asymptotic convergence near isolated singularities of geometric extrema*, Indiana Univ. Math. J. **37** (1988), 225–254. MR 963501 (90b:58046)

[3] W. K. Allard and F. J. Almgren, Jr., *On the radial behavior of minimal surfaces and the uniqueness of their tangent cones*, Ann. of Math. (2) **113** (1981), no. 2, 215–265. MR 607893

[4] D. D. Ang and V. T. Tuan, *An elementary proof of the Morse-Palais lemma for Banach spaces*, Proc. Amer. Math. Soc. **39** (1973), 642–644. MR 0319225

[5] P. Antoine, *Lemme de Morse et calcul des variations*, Bull. Soc. Math. France Mém. (1979), no. 60, 25–29, Analyse non convexe (Proc. Colloq., Pau, 1977). MR 562253

[6] V. I. Arnold, S. M. Gusein-Zade, and A. N. Varchenko, *Singularities of differentiable maps. Volume 1*, Modern Birkhäuser Classics, Birkhäuser/Springer, New York, 2012, Classification of critical points, caustics and wave fronts, Translated from the Russian by Ian Porteous based on a previous translation by Mark Reynolds, Reprint of the 1985 edition. MR 2896292

[7] D. M. Austin and P. J. Braam, *Morse-Bott theory and equivariant cohomology*, The Floer memorial volume, Progr. Math., vol. 133, Birkhäuser, Basel, 1995, pp. 123–183. MR 1362827 (96i:57037)

[8] A. Banyaga and D. E. Hurtubise, *A proof of the Morse–Bott lemma*, Expo. Math. **22** (2004), no. 4, 365–373. MR 2075744

[9] M. Berger, *Nonlinearity and functional analysis*, Academic Press, New York, 1977. MR 0488101 (58 #7671)

[10] E. Bierstone and P. D. Milman, *Semianalytic and subanalytic sets*, Inst. Hautes Études Sci. Publ. Math. (1988), no. 67, 5–42. MR 972342 (89k:32011)

[11] ———, *Canonical desingularization in characteristic zero by blowing up the maximum strata of a local invariant*, Invent. Math. **128** (1997), no. 2, 207–302. MR 1440306

[12] J. Bolton and L. Fernández, *On the regularity of the space of harmonic 2-spheres in the 4-sphere*, Harmonic maps and differential geometry, Contemp. Math., vol. 542, Amer. Math. Soc., Providence, RI, 2011, pp. 187–194. MR 2796649

[13] R. Bott, *Nondegenerate critical manifolds*, Ann. of Math. (2) **60** (1954), 248–261. MR 0064399 (16,276f)

[14] ———, *The stable homotopy of the classical groups*, Ann. of Math. (2) **70** (1959), 313–337. MR 0110104

[15] H. Brézis, *Functional analysis, Sobolev spaces and partial differential equations*, Universitext, Springer, New York, 2011. MR 2759829 (2012a:35002)

[16] Th. Bröcker, *Differentiable germs and catastrophes*, Cambridge University Press, Cambridge-New York-Melbourne, 1975, Translated from the German, last chapter and bibliography by L. Lander, London Mathematical Society Lecture Note Series, No. 17. MR 0494220

[17] A. Cambini, *Sul lemma di M. Morse*, Boll. Un. Mat. Ital. (4) **7** (1973), 87–93. MR 0315738

[18] O. Caps, *Evolution equations in scales of Banach spaces*, Teubner-Texte zur Mathematik [Teubner Texts in Mathematics], vol. 140, B. G. Teubner, Stuttgart, 2002. MR 2352809 (2008m:34129)

[19] A. Carlotto, O. Chodosh, and Y. A. Rubinstein, *Slowly converging Yamabe flows*, Geom. Topol. **19** (2015), no. 3, 1523–1568, arXiv:1401.3738. MR 3352243

[20] R. Chill, *On the Łojasiewicz–Simon gradient inequality*, J. Funct. Anal. **201** (2003), 572–601. MR 1986700 (2005c:26019)

[21] T. H. Colding and W. P. Minicozzi, II, *Łojasiewicz inequalities and applications*, Surveys in Differential Geometry **XIX** (2014), 63–82, arXiv:1402.5087.

[22] ———, *Uniqueness of blowups and Łojasiewicz inequalities*, Ann. of Math. (2) **182** (2015), no. 1, 221–285. MR 3374960

[23] T. H. Colding, W. P. Minicozzi, II, and E. K. Pedersen, *Mean curvature flow*, Bull. Amer. Math. Soc. (N.S.) **52** (2015), no. 2, 297–333. MR 3312634

[24] T. A. Crawford, *The space of harmonic maps from the* 2*-sphere to the complex projective plane*, Canad. Math. Bull. **40** (1997), no. 3, 285–295. MR 1464837

[25] K. Deimling, *Nonlinear functional analysis*, Springer–Verlag, Berlin, 1985. MR 787404 (86j:47001)

[26] S. K. Donaldson and P. B. Kronheimer, *The geometry of four-manifolds*, Oxford University Press, New York, 1990. MR 1079726 (92a:57036)

[27] J. Eells and L. Lemaire, *Selected topics in harmonic maps*, CBMS Regional Conference Series in Mathematics, vol. 50, American Mathematical Society, Providence, RI, 1983. MR 703510 (85g:58030)

[28] Math Stack Exchange, *The dual of the direct sum*, Internet, October 20, 2012, `https://math.stackexchange.com/questions/217596/the-dual-of-the-direct-sum`.

[29] P. M. N. Feehan, *Global existence and convergence of solutions to gradient systems and applications to Yang–Mills gradient flow*, arXiv:1409.1525v4, xx+475 pages.

[30] ———, *Optimal Łojasiewicz–Simon inequalities and Morse–Bott Yang–Mills energy functions*, arXiv:1706.09349.

[31] ———, *Resolution of singularities and geometric proofs of the Łojasiewicz inequalities*, arXiv:1708.09775.

[32] P. M. N. Feehan and M. Maridakis, *Łojasiewicz–Simon gradient inequalities for analytic and Morse–Bott functions on Banach spaces and applications to harmonic maps*, arXiv:1510.03817v6.

[33] ———, *Łojasiewicz–Simon gradient inequalities for analytic and Morse–Bott functions on Banach spaces and applications to harmonic maps*, arXiv:1510.03817v5.

[34] L. Fernández, *The dimension and structure of the space of harmonic 2-spheres in the m-sphere*, Ann. of Math. (2) **175** (2012), no. 3, 1093–1125. MR 2912703

[35] W. M. Goldman and J. J. Millson, *The deformation theory of representations of fundamental groups of compact Kähler manifolds*, Inst. Hautes Études Sci. Publ. Math. (1988), no. 67, 43–96. MR 972343

[36] D. Gromoll and W. Meyer, *On differentiable functions with isolated critical points*, Topology **8** (1969), 361–369. MR 0246329

[37] V. Guillemin and S. Sternberg, *Geometric asymptotics*, American Mathematical Society, Providence, R.I., 1977, Mathematical Surveys, No. 14. MR 0516965

[38] S. Guo and J. Wu, *Bifurcation theory of functional differential equations*, Applied Mathematical Sciences, vol. 184, Springer, New York, 2013. MR 3098815

[39] A. Haraux, *Positively homogeneous functions and the Łojasiewicz gradient inequality*, Ann. Polon. Math. **87** (2005), 165–174. MR 2208543

[40] F. Hélein, *Harmonic maps, conservation laws and moving frames*, second ed., Cambridge Tracts in Mathematics, vol. 150, Cambridge University Press, 2002. MR 1913803 (2003g:58024)

[41] D. Henry, *Geometric theory of semilinear parabolic equations*, Lecture Notes in Mathematics, vol. 840, Springer–Verlag, Berlin-New York, 1981. MR 610244 (83j:35084)

[42] H. Hironaka, *Resolution of singularities of an algebraic variety over a field of characteristic zero. I, II*, Ann. of Math. (2) **79** (1964), 109–203; ibid. (2) **79** (1964), 205–326. MR 0199184

[43] H. Hofer, *A note on the topological degree at a critical point of mountainpass-type*, Proc. Amer. Math. Soc. **90** (1984), no. 2, 309–315. MR 727256

[44] ———, *The topological degree at a critical point of mountain-pass type*, Nonlinear functional analysis and its applications, Part 1 (Berkeley, Calif., 1983), Proc. Sympos. Pure Math., vol. 45, Amer. Math. Soc., Providence, RI, 1986, pp. 501–509. MR 843584

[45] L. Hörmander, *The analysis of linear partial differential operators, III. Pseudo-differential operators*, Springer, Berlin, 2007. MR 2304165 (2007k:35006)

[46] R. A. Horn and C. R. Johnson, *Topics in matrix analysis*, Cambridge University Press, Cambridge, 1994, Corrected reprint of the 1991 original. MR 1288752 (95c:15001)

[47] S.-Z. Huang, *Gradient inequalities*, Mathematical Surveys and Monographs, vol. 126, American Mathematical Society, Providence, RI, 2006. MR 2226672 (2007b:35035)

[48] J. Jost, *Riemannian geometry and geometric analysis*, sixth ed., Universitext, Springer, Heidelberg, 2011. MR 2829653

[49] G. Krantz, S and H. R. Parks, *A primer of real analytic functions*, second ed., Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks], Birkhäuser Boston, Inc., Boston, MA, 2002. MR 1916029

[50] N. H. Kuiper, *C^1-equivalence of functions near isolated critical points*, (1972), 199–218. Ann. of Math. Studies, No. 69. MR 0413161

[51] H. H. Kuo, *The Morse-Palais lemma on Banach spaces*, Bull. Amer. Math. Soc. **80** (1974), 363–365. MR 0334274

[52] H. Kwon, *Asymptotic convergence of harmonic map heat flow*, Ph.D. thesis, Stanford University, Palo Alto, CA, 2002. MR 2703296

[53] S. Lang, *Fundamentals of differential geometry*, Graduate Texts in Mathematics, vol. 191, Springer–Verlag, New York, 1999. MR 1666820

[54] L. Lemaire and J. C. Wood, *Jacobi fields along harmonic 2-spheres in $\mathbb{C}\mathrm{P}^2$ are integrable*, J. London Math. Soc. (2) **66** (2002), no. 2, 468–486. MR 1920415 (2003k:58022)

[55] ———, *Jacobi fields along harmonic 2-spheres in 3- and 4-spheres are not all integrable*, Tohoku Math. J. (2) **61** (2009), no. 2, 165–204. MR 2541404 (2010g:53117)

[56] J. Lindenstrauss and L. Tzafriri, *On the complemented subspaces problem*, Israel J. Math. **9** (1971), 263–269. MR 0276734

[57] Q. Liu and Y. Yang, *Rigidity of the harmonic map heat flow from the sphere to compact Kähler manifolds*, Ark. Mat. **48** (2010), 121–130. MR 2594589 (2011a:53066)

[58] S. Łojasiewicz, *Ensembles semi-analytiques*, (1965), Publ. Inst. Hautes Etudes Sci., Bures-sur-Yvette. LaTeX version by M. Coste, August 29, 2006 based on mimeographed course notes by S. Łojasiewicz, available at `perso.univ-rennes1.fr/michel.coste/Lojasiewicz.pdf`.

[59] J. Mawhin and M. Willem, *On the generalized Morse lemma*, Bull. Soc. Math. Belg. Sér. B **37** (1985), no. 2, 23–29. MR 845402

[60] J. W. Milnor, *Dynamics in one complex variable*, third ed., Annals of Mathematics Studies, vol. 160, Princeton University Press, Princeton, NJ, 2006. MR 2193309

[61] J. D. Moore, *Introduction to global analysis*, Graduate Studies in Mathematics, vol. 187, American Mathematical Society, Providence, RI, 2017, Minimal surfaces in Riemannian manifolds. MR 3729450

[62] J. Moser, *On the volume elements on a manifold*, Trans. Amer. Math. Soc. **120** (1965), 286–294. MR 0182927

[63] L. I. Nicolaescu, *An invitation to Morse theory*, second ed., Universitext, Springer, New York, 2011. MR 2883440 (2012i:58007)

[64] L. Nirenberg, *Topics in nonlinear functional analysis*, Courant Lecture Notes in Mathematics, vol. 6, New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 2001, Chapter 6 by E. Zehnder, Notes by R. A. Artino, Revised reprint of the 1974 original. MR 1850453

[65] R. S. Palais, *Morse theory on Hilbert manifolds*, Topology **2** (1963), 299–340. MR 0158410 (28 #1633)

[66] ———, *The Morse lemma for Banach spaces*, Bull. Amer. Math. Soc. **75** (1969), 968–971. MR 0253378

[67] A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, Applied Mathematical Sciences, vol. 44, Springer–Verlag, New York, 1983. MR 710486 (85g:47061)

[68] M. Petro, *Moduli spaces of Riemann surfaces*, Ph.D. thesis, The University of Wisconsin–Madison, Madison, WI, 2008, p. 119. MR 2712204

[69] A. Pietsch, *Operator ideals*, North-Holland Mathematical Library, vol. 20, North-Holland Publishing Co., Amsterdam-New York, 1980, Translated from German by the author. MR 582655

[70] M. M. Postnikov and Yu. B. Rudyak, *Morse lemma*, Encyclopedia of Mathematics, Springer Verlag and the European Mathematical Society, `http://www.encyclopediaofmath.org/index.php?title=Morse_lemma&oldid=32324`.

[71] T. Poston and I. Stewart, *Catastrophe theory and its applications*, Dover Publications, Inc., Mineola, NY, 1996, With an appendix by D. R. Olsen, S. R. Carter and A. Rockwood, Reprint of the 1978 original. MR 1426131

[72] J. Råde, *On the Yang–Mills heat equation in two and three dimensions*, J. Reine Angew. Math. **431** (1992), 123–163. MR 1179335 (94a:58041)

[73] W. Rudin, *Functional analysis*, second ed., International Series in Pure and Applied Mathematics, McGraw-Hill, Inc., New York, 1991. MR 1157815

[74] P. Seidel, *A long exact sequence for symplectic Floer cohomology*, Topology **42** (2003), no. 5, 1003–1063. MR 1978046

[75] G. R. Sell and Y. You, *Dynamics of evolutionary equations*, Applied Mathematical Sciences, vol. 143, Springer, New York, 2002. MR MR1873467 (2003f:37001b)

[76] L. Simon, *Asymptotics for a class of nonlinear evolution equations, with applications to geometric problems*, Ann. of Math. (2) **118** (1983), 525–571. MR 727703 (85b:58121)

[77] ______, *Isolated singularities of extrema of geometric variational problems*, Lecture Notes in Math., vol. 1161, Springer, Berlin, 1985. MR 821971 (87d:58045)

[78] ______, *Theorems on regularity and singularity of energy minimizing maps*, Lectures in Mathematics ETH Zürich, Birkhäuser, Basel, 1996. MR 1399562 (98c:58042)

[79] C. T. Simpson, *Higgs bundles and local systems*, Inst. Hautes Études Sci. Publ. Math. (1992), no. 75, 5–95. MR 1179076

[80] S. Smale, *Morse theory and a non-linear generalization of the Dirichlet problem*, Ann. of Math. (2) **80** (1964), 382–396. MR 0165539 (29 #2820)

[81] M. Struwe, *Variational methods*, fourth ed., Springer, Berlin, 2008. MR 2431434 (2009g:49002)

[82] H. Tanabe, *Equations of evolution*, Monographs and Studies in Mathematics, vol. 6, Pitman (Advanced Publishing Program), Boston, MA, 1979, Translated from the Japanese by N. Mugibayashi and H. Haneda. MR 533824 (82g:47032)

[83] ______, *Functional analytic methods for partial differential equations*, Monographs and Textbooks in Pure and Applied Mathematics, vol. 204, Marcel Dekker Inc., New York, 1997. MR 1413304 (97i:35002)

[84] P. Topping, *Rigidity in the harmonic map heat flow*, J. Differential Geom. **45** (1997), 593–610. MR 1472890 (99d:58050)

[85] A. J. Tromba, *The Morse lemma on Banach spaces*, Proc. Amer. Math. Soc. **34** (1972), 396–402. MR 0295395

[86] V. T. Tuan and D. D. Ang, *A representation theorem for differentiable functions*, Proc. Amer. Math. Soc. **75** (1979), no. 2, 343–350. MR 532164

[87] K. K. Uhlenbeck, *Morse theory on Banach manifolds*, Bull. Amer. Math. Soc. **76** (1970), 105–106. MR 0253381

[88] ______, *The Chern classes of Sobolev connections*, Comm. Math. Phys. **101** (1985), 449–457. MR 815194 (87f:58028)

[89] A. Weinstein, *Symplectic structures on Banach manifolds*, Bull. Amer. Math. Soc. **75** (1969), 1040–1041. MR 0245052

[90] E. F. Whittlesey, *Analytic functions in Banach spaces*, Proc. Amer. Math. Soc. **16** (1965), 1077–1083. MR 0184092 (32 #1566)

[91] A. Yagi, *Abstract parabolic evolution equations and their applications*, Springer Monographs in Mathematics, Springer–Verlag, Berlin, 2010. MR 2573296 (2011c:35008)

[92] E. Zeidler, *Nonlinear functional analysis and its applications, I. Fixed-point theorems*, Springer, New York, 1986. MR 816732 (87f:47083)

Department of Mathematics, Rutgers, The State University of New Jersey, 110 Frelinghuysen Road, Piscataway, NJ 08854-8019, United States of America

E-mail address: feehan@math.rutgers.edu